1. Given a random sample \{Y_1, \ldots, Y_n\}, show that \( S^2 \) and \( \tilde{S}^2 \) are unbiased estimators for the population variance \( \sigma^2 \).

\[
S^2 = \frac{1}{n} \sum_{i=1}^{n} (Y_i - \mu)^2, \quad \tilde{S}^2 = \frac{1}{n-1} \sum_{i=1}^{n} (Y_i - \bar{Y})^2
\]

Note that \( \mu = E(Y) \) and \( \bar{Y} = \frac{1}{n} \sum_{i=1}^{n} Y_i \).

2. At a party, \( n \) men take off their hats. The hats are then mixed up and each man randomly selects one. We say that a match occurs if a man selects his own hat. What is the probability of no matches? What is the probability of exactly \( k \) matches?

3. If \( X \) and \( Y \) are identically distributed, not necessarily independent, show that \( \text{Cov}(X + Y, X - Y) = 0 \)

4. Let \( X_1, X_2, \ldots, X_n \) denote a sample from a population whose mean value \( \theta \) is unknown. Argue that among all unbiased estimators of \( \theta \) of the form \( \sum_{i=1}^{n} \lambda_i X_i \) with \( \sum_{i=1}^{n} \lambda_i = 1 \) the one with minimum mean square error has \( \lambda_i = \frac{1}{n} \), \( i=1,\ldots,n \).

5. Let \( X_1, X_2, \ldots, X_n \) be i.i.d Bernoulli random variables, with unknown parameter \( p \in (0,1) \). The aim of this exercise is to estimate the common variance of the \( X_i \)'s.

   (a) Show that \( \text{var}(X_i) = p.(1 - p) \)

   (b) Compute the bias of this estimator.

   (c) Find an unbiased estimator of \( p(1 - p) \).

6. Assume \( X \) and \( Y \) follow a joint normal distribution:

\[
f_{XY}(x, y) = \frac{1}{2\pi \sqrt{\det(\Sigma)}} \exp \left[ -\frac{1}{2} \left( \begin{array}{c} x - \mu_x \\ y - \mu_y \end{array} \right) \Sigma^{-1} \left( \begin{array}{c} x - \mu_x \\ y - \mu_y \end{array} \right) \right]
\]
where \( \Sigma = \begin{pmatrix} \sigma_x^2 & \rho \sigma_x \sigma_y \\ \rho \sigma_x \sigma_y & \sigma_y^2 \end{pmatrix} \) is the variance covariance matrix of the distribution, \( \mu_x \) and \( \mu_y \) are mean of marginal distributions of \( X \) and \( Y \) respectively.

(a) What does the parameter \( \rho \) show? What values do you expect this to take?
(b) Prove that if \( \text{cov}(X,Y) = 0 \) then \( X \) and \( Y \) are independent.
(c) Find \( f_{Y \mid X}(y \mid x) \).
(d) What is \( E(Y \mid X = x) \)? If we are only interested in finding \( E(Y \mid X) \) how many parameters do we need to estimate?