Motivation

• How should we model individual decision making?
• Agents have preferences over different alternatives.
  ■ What do agents like most?
• The feasible set of alternatives determines possible choices.
  ■ What can agents choose?
  ■ The institutional setting shapes the feasible set.
Contents

- Consumer preferences and their properties
- Utility function
- Existence of a utility function
- Utility maximization $\rightarrow$ demand functions $\rightarrow$ indirect utility
  - When does demand satisfy HD0 and Walras’ law?
- Expenditure minimization $\rightarrow$ demand functions $\rightarrow$ expenditure function
- Relation between the two problems
- Welfare measurement
Outline

Introduction

Basic elements

Utility maximization problem

Expenditure Minimization Problem

Comparative Statics

UMP vs. EMP

Welfare measurement
Consumption Set – Feasibility

- Consumption set ($X$): a subset of the commodity space a consumer could possibly consume given *physical constraints* from the environment.
  - consumption of non-negative values of food and water
  - consumption of leisure and food
  - survival needs
- $X = \mathbb{R}_+^L = \{\mathbf{x} \in \mathbb{R}^L : x_\ell \geq 0 \text{ for } \ell = 1, \ldots, L\}$
  - convex set: if $\mathbf{x}, \mathbf{x}' \in X$ then $\mathbf{x}'' = \alpha \mathbf{x} + (1 - \alpha) \mathbf{x}' \in X$ for all $\alpha \in [0, 1]$
- Is a consumption set with integer values for commodities convex?
Preference Relations

- $X$: Consumption set (set of consumption bundles)
- Preference relation ($\succeq$) denotes the comparison of two bundles $x, y \in X$
  - Read $x \succeq y$ as “$x$ is at least as good as $y$”
- Strict preference relation ($\succ$) is defined as
  $x \succ y \iff x \succeq y$ but not $y \succeq x$
  - Read $x \succ y$ as “$x$ is strictly preferred to $y$”
- Indifference relation ($\sim$) is defined as
  $x \sim y \iff x \succeq y$ and $y \succeq x$
Indifference curves

- For any $y \in X$, define $IC(y) = \{x \in X \mid x \sim y\}$
- Often a very useful tool for graphical representation of preferences.
- Not that useful with more than two dimensions.
- Define, upper contour curve

$$UC(y) = \{x \in X \mid x \succsim y\}$$

- Similarly define lower contour curve

$$LC(y) = \{x \in X \mid y \succsim x\}$$
Rationality

• $\simeq$ is rational if

1. Complete: allows comparison of ALL alternatives
   for all $x, y \in X$ we have either $x \simeq y$ or $y \simeq x$ (or both).
2. Transitive: no cycles
   for all $x, y, z \in X$, if $x \simeq y$ and $y \simeq z$, then $x \simeq z$.

• Both are strong assumptions
  - Is completeness the same as indifference?
  - Does transitivity feel natural?
Desirability

- Monotone (M): If $x, y \in X$ and $y \gg x$ implies $y \succ x$.
  - More of everything is better.

- Strongly monotone (SM): If $y \geq x$ and $y \neq x$ implies $y \succ x$.
  - More of at least one thing is better.

- Locally nonsatiated (LNS): If for $\forall x \in X$ and $\forall \epsilon > 0$, there is $y \in X$ such that $\| y - x \| \leq \epsilon$ and $y \succ x$.
  - In all possible neighborhoods of all bundles there exists a better option!

- Strong monotonicity $\Rightarrow$ Monotonicity $\Rightarrow$ Local Nonsatiation
  - Proof exercise.
Illustration of desirability

- Graphs!
- Does LNS imply more is better?
- What types of IC are not compatible with LNS?
  - thick ICs, all commodities are bads!
Convexity

- Convex: If $\forall x \in X$, $UC(x)$ is a convex set.
  - OR If $y \succeq x$ and $z \succeq x$, then $\alpha y + (1 - \alpha)z \succeq x$ for any $\alpha \in [0, 1]$.
  - Implies diminishing marginal rate of substitution.
  - Agents like diversification. E.g. If $x \sim y$ then $\frac{1}{2}x + \frac{1}{2}y \succeq x$ and $\frac{1}{2}x + \frac{1}{2}y \succeq y$.

- Strictly convex: If $y \succeq x$, $z \succeq x$, and $y \neq z$ then $\alpha y + (1 - \alpha)z \succ x$ for any $\alpha \in (0, 1)$.
  - If $x \sim y$ then which of these is right? a) $\alpha y + (1 - \alpha)x \succ x$
    b) $\alpha y + (1 - \alpha)x \sim x$

- Can you think of a situation where diversification is NOT preferred?
Continuity

- Continuous: if $UC(x)$ and $LC(x)$ are both closed sets for all $x \in X$.
  
  - What is a closed set?
  - What type of behavior is ruled out when preferences are continuous?

- Example: Lexicographic preferences
Homotheticity

- Definition: A monotone \( \succeq \) on \( X = \mathbb{R}_+^L \) is homothetic if all indifference sets are related by proportional expansion along rays.
- OR if \( x \sim y \) then \( \alpha x \sim \alpha y \) for any \( \alpha \geq 0 \).
- Think about the shape of ICs.
  - What types of preferences are ruled out when we assume homotheticity?
Quasilinearity

- Definition: $\succsim$ on $X = (-\infty, \infty) \times \mathbb{R}_{+}^{L-1}$ is quasilinear w.r.t. commodity 1 (the numeraire) if
  - Indifference curves are parallel shifts of each other along the axis of commodity 1.
    - OR if $x \sim y$ then $(x + \alpha e_1) \sim (y + \alpha e_1)$ for $e_1 = (1, 0, \ldots, 0)$ and any $\alpha \in \mathbb{R}$.
  - Good 1 is desirable: $x + \alpha e_1 \succ x$ for all $x$ and $\alpha > 0$.

- Shape of ICs.
- We can deduce the entire preference relation from one IC if $\succsim$ is quasilinear or homothetic.
Utility Function

- Utility function: $u(x)$ assigns a numerical value to each element of $X$ ($u : X \rightarrow \mathbb{R}$)
- $u(x)$ represents the preference relation $\succeq$ if for all $x, y \in X$,

$$x \succeq y \iff u(x) \geq u(y)$$

- Assume, there is a $u(x)$ that represents $\succeq$, could you find another utility function that does the same?
Could We Express $\succeq$ by $u(x)$?

- Much easier to work with a utility function than preference relations.
- Rationality, monotonicity, and convexity are not sufficient for existence of $u(x)$.
- Example: Lexicographic preference relation.
  - Take $X = \mathbb{R}^2_+$ and define $x \succ y$ if “$x_1 > y_1$” or “$x_1 = y_1$ and $x_2 > y_2$”
  - This preference relation is rational, strongly monotone, and strictly convex.
  - No utility function represents this!
Existence of $u(x)$

- **Proposition**: If $\succeq$ satisfies *rationality* and *continuity* then there exists a *continuous* utility function $u(x)$ that represents $\succeq$.

  - We prove an easier version of the proposition with the assumption of strong monotonicity. The proof is given in p. 97 of Varian.
Notes on $u(x)$

- $u(\cdot)$ is not unique; any strictly increasing transformation will do.
- Not all $u(\cdot)$ representing a continuous $\succsim$ are continuous. Why?
- We usually assume $u(\cdot)$ to be differentiable. Is continuity of $\succsim$ enough for this. Example?
  - Sometimes, assume $u(\cdot)$ is twice continuously differentiable! $IC(x)$ should be nice and smooth!
**IC(\( \mathbf{x} \)) and Contours of \( u(\mathbf{x}) \)**

- With \( u(\mathbf{x}) \) we can derive the \( IC(\mathbf{x}) \) easily by plotting the contours (level curves) of \( u(\mathbf{x}) \)
- Example: \( u(x_1, x_2) = Ax_1^\alpha x_2^{1-\alpha} \)
  - Level curves:
    
    \[
    \bar{u} = Ax_1^\alpha x_2^{1-\alpha}
    \]
    
    \[
    \Rightarrow x_2 = \left( \frac{\bar{u}}{Ax_1^\alpha} \right)^{\frac{1}{1-\alpha}}
    \]
    
    \[
    \Rightarrow x_2 = cx_1^{-\frac{\alpha}{1-\alpha}}
    \]

  - This gives the loci of all points that deliver \( \bar{u} \).
Properties of $\succsim \Rightarrow$ Properties of $u(x)$

- Monotonicity $\Rightarrow u(x)$ is increasing; i.e. If $x \succsim y$ then $u(x) > u(y)$.
- (Strict) convexity of $\succsim$ $\Rightarrow u(x)$ is (strictly) quasiconcave; i.e. $\forall x, y \in X$ and any $\alpha \in [0, 1]$ we have $u(\alpha x + (1 - \alpha)y) \geq \min\{u(x), u(y)\}$
- A continuous $\succsim$ is homothetic iff it admits a HD1 $u(x)$.
- A continuous $\succsim$ on $(-\infty, \infty) \times \mathbb{R}_+^{L-1}$ is quasilinear w.r.t. the first commodity iff it admits $u(x) = x_1 + \phi(x_2, \ldots, x_L)$.
- Note: monotonicity and convexity are ordinal properties while homotheticity and quasilinearity are cardinal.
Affordability

- Consumption limited to bundles that the consumer can afford.
  - depends on prices and wealth (income)

- Assumptions
  - Principle of completeness: All commodities are traded at publicly known prices.
    - Price vector
      \[ p = \begin{bmatrix} p_1 \\ \vdots \\ p_L \end{bmatrix} \in \mathbb{R}^L \]
    - \( p \gg 0 \); interpretation of negative \( p_\ell \)?
  - Price-taking: prices are beyond the influence of the consumer.
Competitive Budget Sets

- Competitive (Walrasian) budget set:
  \[ B_{p,w} = \{ x \in \mathbb{R}_+^L : p \cdot x \leq w \} \]
  - \( \cdot \) is the inner product of two vectors OR a short hand notation for \( \sum_{\ell=1}^{L} p_\ell x_\ell \)
  - Restate consumer’s problem: choose a consumption bundle from \( B_{p,w} \)
  - Is this convex?

- Upper boundary of budget set OR budget hyperplane (line)
  \[ \{ x \in \mathbb{R}^L : p \cdot x = w \} \]

- Graphical representation when \( L = 2 \).
  - slope of the budget line: \( -(p_1/p_2) \) reflects relative terms of exchange between commodities
  - reduction/increase in one/both prices
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Utility Maximization Problem

- Assume $\succsim$ is rational, continuous, and LNS. Take the continuous $u(x)$ representing this preference relation.
- Consumer’s decision problem

$$\max_{x \geq 0} u(x)$$

$$\text{s.t.} \quad p \cdot x \leq w$$

is now a utility maximization problem (UMP) $[p \gg 0, \ w > 0, \ X = \mathbb{R}^L_+]$.

- Pick the consumption bundle from the Walrasian budget set that gives the highest level of utility!
- Walrasian demand: optimal consumption bundle, solution to UMP
- Indirect utility: maximal utility value
Existence of Solution

- If \( p \gg 0 \) and \( u(x) \) is continuous, then UMP has a solution.
- Proof: If \( p \gg 0 \) then \( B_{p,w} = \{ x \in \mathbb{R}_+^L : p \cdot x \leq w \} \) is a compact set [is bounded and closed].
  A famous result says that a continuous function always has a max (and min) on a compact set.
- If one price is zero (e.g. \( p_\ell = 0 \)) can we still make the argument above?
How to Find the Solution?

- Kuhn-Tucker conditions
  - often simplified by utilizing features of the utility functions
- Graphical solution
- Graphical solution may not be enough on its own, but is extremely helpful in delivering the intuition.
Walrasian Demand

• Demand is

\[ x(p, w) = \arg\max_{x \geq 0} u(x) \]
\[ \text{s.t. } p \cdot x \leq w \]

• This could be multi/single-valued.

• Basically, find the points that fall on the highest IC that has an intersection with \( B_{p,w} \).
Properties of Demand

- Suppose \( u(x) \) is a continuous utility function representing \( \succsim \) with LNS and \( X = \mathbb{R}_+^L \), then the demand correspondence has the following properties:

1. HD0 in \((p, w)\), i.e. \( x(\alpha p, \alpha w) = x(p, w) \) for any \( p, w \), and \( \alpha > 0 \).
2. Walras’ law: \( p \cdot x = w \) for all \( x \in x(p, w) \).
3. Convexity: if \( \succsim \) convex \((u \text{ quasiconcave})\), then \( x(p, w) \) is a convex set.
4. Uniqueness: if \( \succsim \) strictly convex \((u \text{ strictly quasiconcave})\), then \( x(p, w) \) is single-valued.
5. Continuous in \((p, w)\).
Kuhn-Tucker Necessary Conditions

- If $u(x)$ continuously differentiable, then use calculus to characterize the solution.
- Kuhn-Tucker (KT) necessary conditions: If $x^*$ is a solution to UMP, then it satisfies the following properties
  1. $\exists \lambda \geq 0$ such that for all $\ell = 1, \ldots, L$:
     $$\frac{\partial u(x^*)}{\partial x_\ell} \leq \lambda p_\ell, \text{ with equality if } x^*_\ell > 0$$
  2. Complementary slackness: $\lambda (p_1 x^*_1 + \cdots + p_k x^*_k - w) = 0$
     With LNS this condition is simplified to $p \cdot x^* = w$
Sufficient Conditions

- When are the necessary conditions also sufficient?
  - When can we say that an $x^*$ that satisfies KT necessary conditions is a maximizer?

- The second-order condition can be written as:

$$h^t D^2 u(x^*) h \leq 0 \quad \text{for all } h \text{ such that } p \cdot h = 0$$

- This requires the Hessian matrix to be NSD for all $h$ orthogonal to the price vector.
What Convexity Buys!

- Theorem: If $u(x)$ is quasiconcave and the constraint set is convex, then the KT necessary conditions are sufficient.

- Notes:
  - $\succeq$ convex $\Rightarrow u$ quasiconcave.
  - $\{(x_1, \ldots, x_L) \geq 0 | p_1 x_1 + \cdots + p_L x_L \leq w\}$ is a convex set.
  - With strict quasiconcavity we can show the solution is unique.
What LNS Buys!

- With LNS, the budget constraint has to bind, i.e. \( \mathbf{p} \cdot \mathbf{x} = w \), therefore (with caution) you can use Lagrangian instead of KT conditions.

- Notice we have assumed \( u \) is continuously differentiable. You need to check non-differentiable points separately.
  - The safest way is to plug in the solutions of Lagrangian/KT conditions together with boundary and non-differentiable points and see which one delivers the largest value.
Interior Solution

- Interior solution is when \( x^* \gg 0 \).
- At any interior solution of UMP we have \( \frac{\partial u(x^*)}{\partial x_\ell} = \lambda p_\ell \)
  - i.e. gradient of \( u \) at \( x^* \) is in the same direction as the price vector (\( \lambda > 0 \)).
  - Graphical illustration?

- Consider the F.O.C.s for commodities \( \ell \) and \( k \) and divide them

\[
MRS_{\ell k}(x^*) \equiv \frac{\frac{\partial u(x^*)}{\partial x_\ell}}{\frac{\partial u(x^*)}{\partial x_k}} = \frac{p_\ell}{p_k}
\]

- At the interior solution: marginal rate of substitution of good \( \ell \) for \( k \) (willingness) equals price ratio (affordability).

- We can derive the above condition using the perturbation argument: say for an interior solution \( MRS_{\ell k}(x^*) > \frac{p_\ell}{p_k} \) then exchange \( x_\ell \) for \( x_k \) to reach higher utility...
Marginal Rate of Substitution

- $MRS$ shows the exchanges that leave the consumer on the same indifference curve.
- In fact from its definition

$$MRS_{\ell k}(x^*) = \frac{MU_\ell(x^*)}{MU_k(x^*)} = \frac{\partial u(x^*)}{\partial x_\ell} \frac{\partial u(x^*)}{\partial x_k}$$

- Taking $dx_\ell$ units of good $\ell$ reduces utility by $MU_\ell(x^*)dx_\ell$ giving $dx_\ell = \frac{MU_\ell(x^*)dx_\ell}{MU_k(x^*)} = MRS_{\ell k}(x^*)dx_\ell$ will exactly offset this and leaves the consumer on the same indifference curve.
- Convexity results in a diminishing $MRS$.
- This is different from decreasing $MU_\ell$ or $MU_k$. 
Corner Solution

- This is in contrast to an interior solution, i.e. at least one of \( x_\ell^* = 0 \).
- KT necessary conditions state that if \( x_\ell^* = 0 \) then
  \[
  \frac{\partial u(x^*)}{\partial x_\ell} \leq \lambda p_\ell.
  \]
- \( MRS_{\ell k}(x^*) \) is not necessarily equal to the price ratio \( \frac{p_\ell}{p_k} \).
- Notice Walras law could still hold at the corner solution.
Example (Cobb-Douglas)

- Cobb-Douglas utility:

\[
\max_{(x_1, x_2) \geq 0} u(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}
\]

\[
s.t. \quad p_1 x_1 + p_2 x_2 \leq w
\]

- Solving the UMP yields

\[
x_1 = \alpha \frac{w}{p_1}
\]

\[
x_2 = (1 - \alpha) \frac{w}{p_2}
\]

- Notice KT necessary conditions are sufficient because
  - \( u \) is quasiconcave since \( UC(x) \) is a convex set for all \( x \in \mathbb{R}^2_+ \).
  - The constraint set is convex.
Example (Linear Utility)

- Consider

\[
\max_{(x_1, x_2) \geq 0} u(x_1, x_2) = x_1 + x_2 \\
\text{s.t. } p_1 x_1 + p_2 x_2 \leq w
\]

- \(u\) is quasiconcave (and quasiconvex) and we have the normal BC, therefore KT necessary conditions are sufficient.

- \(u\) is not strictly quasiconcave \(\Rightarrow\) there might be several maximizers.

- Do we have an interior or a corner solution?
  - Graphical illustration helps a lot.
  - Could follow KT.
Example (Quasilinear Utility)

- Consider

$$\begin{align*}
\max_{(x_1, x_2) \geq 0} & \quad u(x_1, x_2) = x_1 + \sqrt{x_2} \\
\text{s.t.} & \quad p_1 x_1 + p_2 x_2 \leq w
\end{align*}$$

- $u$ is strictly quasiconcave and constraint set is convex $\Rightarrow$ KT necessary conditions are sufficient.
  
  - Maximizer is unique but is it at the corner or interior of budget line?
Interpretation of $\lambda$

- $\lambda$ the Lagrange multiplier gives the marginal (shadow) value of relaxing the constraint in UMP

$$u(x(p, w)) = \max_{x \geq 0} u(x) \quad \text{s.t.} \quad p \cdot x \leq w$$

- Relaxing constraint $\rightarrow$ increase $w$

- Therefore, $\lambda$ captures the marginal utility value of increasing wealth (income)!

- Formally

$$\frac{\partial u(x(p, w))}{\partial w} = \nabla u(x) \cdot D_w x(p, w)$$

$$= \lambda p \cdot D_w x(p, w)$$

$$= 1 \text{ by Walras law}$$

$$= \lambda$$
Indirect Utility

- Indirect utility function \( v(p, w) \) is the value of the utility at the optimal consumption bundle

\[
v(p, w) = u(x(p, w)) = \max_{x \geq 0} u(x) \quad \text{s.t.} \quad p \cdot x \leq w
\]

- Obviously this depends on the particular \( u(x) \) chosen (in contrast to \( x(p, w) \)).
Examples

- Cob-Douglas utility gave \( x_1 = \alpha \frac{w}{p_1} \) and \( x_2 = (1 - \alpha) \frac{w}{p_2} \). Insert these into the original utility function to get
  
  ■ the indirect utility:
  
  \[
  v(p, w) = \left( \frac{\alpha w}{p_1} \right)^{\alpha} \left( \frac{(1-\alpha)w}{p_2} \right)^{1-\alpha} = \frac{\alpha^\alpha (1-\alpha)^{1-\alpha}}{p_1^\alpha p_2^{1-\alpha}} w
  \]

  ■ using \( u = \alpha \ln x_1 + (1 - \alpha) \ln x_2 \) gives exactly the same demand functions but a different indirect utility function.

- Linear utility gives \( v(p, w) = x_1(p, w) + x_2(p, w) \)
• Suppose $u(x)$ is a continuous utility function representing $\succeq$ with LNS and $X = \mathbb{R}_+^L$, then $v(p, w)$ satisfies

1. HD0 in $(p, w)$, i.e. $v(\alpha p, \alpha w) = v(p, w)$ for any $p$, $w$, and $\alpha > 0$.
2. Strictly increasing in $w$ and non-increasing in $p_\ell$ for any $\ell$.
3. Quasiconvex: i.e. $LC(\bar{v}) = \{(p, w) : v(p, w) \leq \bar{v}\}$ is convex for any $\bar{v}$.
4. Continuous in $(p, w)$.
5. Roy’s Identity:

$$x_\ell(p, w) = - \frac{\partial v/\partial p_\ell}{\partial v/\partial w}$$

• Proofs covered (except for continuity)!
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**Expenditure Minimization Problem (EMP)**

- Find the minimal level of wealth required to achieve a given level of utility:

\[
\min_{x \geq 0} \quad p \cdot x \\
\text{s.t.} \quad u(x) \geq u
\]

is the EMP [assumptions: \( p \gg 0 \), \( u > u(0) \), \( X = \mathbb{R}^L_+ \)].

- In comparison to UMP: roles of constraint and objective function are reversed.
  - Under LNS, expenditure function is the inverse of the indirect utility function!

- Both UMP and EMP are considering efficient use of consumer’s purchasing power.

- Expenditure minimizing commodity vectors: \( h(p, u) \)
  - Hicksian (compensated) demand

- Minimal value: \( e(p, u) \) expenditure function
Solution to EMP

\[
\begin{align*}
\min_{\mathbf{x} \geq 0} & \quad \mathbf{p} \cdot \mathbf{x} \\
\text{s.t.} & \quad u(\mathbf{x}) \geq u
\end{align*}
\]

- KT F.O.C. \( \exists \lambda \geq 0 \)

\[
\mathbf{p} \geq \lambda \nabla u(\mathbf{x}^*) \quad \text{with equality in } l\text{th element if } x^{*}_l > 0
\]

Or

\[
p_l \geq \lambda \frac{\partial u(\mathbf{x}^*)}{\partial x^*_l} \quad \text{with equality if } x^{*}_l > 0
\]

- The other KT F.O.C. is

\[
\lambda \cdot [u(\mathbf{x}^*) - u] = 0
\]

which means either \( u(\mathbf{x}^*) = u \) or \( \lambda = 0 \).
Expenditure Function

\[ e(p, u) = \min_{x \geq 0} p \cdot x \]
\[ \text{s.t. } u(x) \geq u \]

- Suppose \( u \) is continuous and represents an LNS preference on \( X = R^L_+ \) and \( p \gg 0 \), then \( e(p, u) \) is
  1. HD1 in \( p \).
  2. strictly increasing in \( u \) and non-decreasing in \( p \)
  3. concave in \( p \)
  4. continuous in \( p \) and \( u \)

- Proofs: we cover 1 and 3; 2 is left as exercise. 4 is not covered.
Concavity of Expenditure Function

- claim: \( e(p'', u) \geq \alpha e(p, u) + (1 - \alpha)e(p', u) \) where \( p'' = \alpha p + (1 - \alpha)p' \)
- Denote the solution to EMP at different price vectors as \( x, x', x'' \)

\[
e(p'', u) = p'' \cdot x'' = \alpha p \cdot x'' + (1 - \alpha)p' \cdot x''
\]

But since \( u(x'') \geq u \) and \( e(p, u) \) is the minimum expenditure for buying \( u \) we have \( p \cdot x'' \geq e(p, u) \) and \( p' \cdot x'' \geq e(p', u) \). Using this in the above expression

\[
e(p'', u) \geq \alpha e(p, u) + (1 - \alpha)e(p', u)
\]

- Intuition: consider \( x \in h(p, u) \),
  - when \( p \) is changed but \( x \) is fixed: expenditure is \( p \cdot x \)
  - when \( p \) is changed and \( x \) is re-optimized: \( e(p, u) \leq p \cdot x \)
Hicksian Demand Properties

- Suppose $u$ is continuous representing an LNS preference on $X = \mathbb{R}^L_+$ and $p \gg 0$, then $h(p, u)$ is
  1. HD0 in $p$
  2. No excess utility: $\forall x \in h(p, u)$ then $u(x) = u$
  3. If $\succ$ convex, then $h(p, u)$ is a convex set.
  4. If $\succ$ strictly convex, then $h(p, u)$ is a singleton.
  5. Shephard’s lemma: \( \frac{\partial e(p, u)}{\partial p_\ell} = h_\ell(p, u) \)

- Proofs covered!
Hicksian (Compensated) Demand

- Why is this called compensated demand?
- \( h(p, u) \) gives the level of demand that would arise if wealth is constantly adjusted to keep the utility at \( u \)
- Hicksian wealth compensation
- Graphical representation
Example - Quasilinear Utility

- Consider \( u(x_1, x_2) = x_1 + \ln x_2 \), the EMP associated with this

\[
\min_{x_1, x_2} \quad p_1 x_1 + p_2 x_2 \\
\text{s.t.} \quad x_1 + \ln x_2 \geq u
\]

- First, notice both the objective function and \( u(\cdot) \) are increasing, therefore the constraint must satisfy with equality.

- F.O.C.

\[
p_1 \geq \lambda \cdot 1 \text{ with equality if } x_1^* > 0
\]

\[
p_2 \geq \lambda \cdot \frac{1}{x_2} \text{ with equality if } x_2^* > 0
\]
Example – Solution

1. If $x_1^* > 0$ and $x_2^* > 0$: $\lambda = p_1$ and $x_2^* = \frac{p_1}{p_2}$, this gives
   
   $x_1 = u - \ln \frac{p_1}{p_2}$ [parametric condition: $u - \ln \frac{p_1}{p_2} > 0$]

2. If $x_1^* = 0$ and $x_2^* > 0$: $x_2^* = e^u$ and $\lambda = p_2 e^u$ [parametric
   condition: $\frac{p_1}{p_2} \geq e^u$]

3. If $x_1^* \geq 0$ and $x_2^* = 0$: constraint non-differentiable. But since
   $\ln x_2 \to -\infty$ as $x_2 \to 0$ the utility constraint cannot
   hold in this case.

   • Notice cases 1 and 2 cover the whole parameter space, i.e. all
   possible values for $p_1$, $p_2$, and $u$. In summary

   \[ h(p_1, p_2, u) = (h_1, h_2) = \begin{cases} 
   (0, e^u) & \text{if } \frac{p_1}{p_2} \geq e^u \\
   (u - \ln \frac{p_1}{p_2}, \frac{p_1}{p_2}) & \text{if } \frac{p_1}{p_2} < e^u 
   \end{cases} \]

   \[ e(p, u) = p \cdot h(p, u) = \begin{cases} 
   p_2 e^u & \text{if } \frac{p_1}{p_2} \geq e^u \\
   p_1 \left(u - \ln \frac{p_1}{p_2} + 1\right) & \text{if } \frac{p_1}{p_2} < e^u 
   \end{cases} \]
Suppose \( u(x) \) continuous represents an LNS \( \succeq \) and \( h(p, u) \) is the Hicksian demand function (single-valued) for all \( p \gg 0 \). Then for all \( p' \) and \( p \)

\[
(p' - p) \cdot [h(p', u) - h(p, u)] \leq 0
\]

Proof: From definition of \( h(p, u) \) we have

\[
\begin{align*}
p' \cdot h(p', u) & \leq p' \cdot h(p, u) \\
p \cdot h(p, u) & \leq p \cdot h(p', u)
\end{align*}
\]

add up to get the result.

Note: \( x(p, w) \) doesn’t show this property. Why?
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Comparative Statics

- We try to examine how the consumer’s demand changes as prices and wealth change.
  - You could also look at changes in expenditure and indirect utility.
- If functions are differentiable, then one can always use derivatives w.r.t. to prices and wealth to get the comparative statics.
  - Graphical representation is, however, extremely useful.
Wealth Expansion Path

- Wealth expansion path (WEP): how does the bundle demanded change as the consumer gets more wealth?
  - This is drawn in the commodity space.
  - Sometimes this is referred to as income expansion path (IEP)
- $\frac{\partial x_\ell(p,w)}{\partial w}$: wealth effect for commodity $\ell$
- Engel function: $x(\bar{p}, w)$
  - WEP is the contour curves of the Engel function
Normal vs. inferior goods

- Normal good: \( \frac{\partial x_\ell(p, w)}{\partial w} \geq 0 \)
  - Luxury: \( \epsilon_{\ell, w} = \frac{\partial x_\ell(p, w)}{\partial w} \times \frac{w}{x_\ell(p, w)} > 1 \)
- Inferior good: \( \frac{\partial x_\ell(p, w)}{\partial w} < 0 \)
Offer curve

- Offer curve for good 1: Fix wealth and the price of other goods, how does demanded bundle vary with $p_1$?
- Two possibilities
  - $p_1 \downarrow \Rightarrow x_1 \uparrow$ vs. $p_1 \downarrow \Rightarrow x_1 \downarrow$ (Giffen)
Example: excise vs income tax

- Tax a consumer to obtain a certain amount of revenue.
  - Consumer’s initial budget constraint: \( p_1 x_1 + p_2 x_2 = w \)
- with sales tax on good 1: \( (p_1 + t)x_1 + p_2 x_2 = w \)
  \[ \Rightarrow \text{after-tax consumption: } (x_1^*, x_2^*), \text{ revenue collected: } tx_1^*. \]
- With income (wealth) tax: \( p_1 x_1 + p_2 x_2 = w - tx_1^*. \)
- Which type of tax leads to a higher utility for the consumer?
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Overview

• We start by relating the solutions to EMP and UMP.
• Then we explore two relationships
  1. Hicksian demand and expenditure function (Shephard’s lemma and some additional results)
  2. Hicksian and Walrasian demand functions (Slutsky equation)
• Previously we discussed Roy’s identity that establishes the link between demand and indirect utility
• When required we rule out multi-valued correspondences.
EMP vs. UMP

• Suppose $u(\cdot)$ is continuous representing locally non-satiated preferences on $X = \mathbb{R}^L_+$ and $p \gg 0$, then

1. If $x^*$ is a solution to UMP when $w > 0$, then $x^*$ is a solution to EMP when $u = u(x^*)$, and $e(p, u) = w$.
2. If $x^*$ is a solution to EMP when $u > u(0)$, then $x^*$ is a solution to UMP when $w = p \cdot x^*$, and $v(p, w) = u(x^*)$.

• This suggests

- $e(p, v(p, w)) = w$ and $v(p, e(p, u)) = u$
- $h(p, u) = x(p, e(p, u))$ and $x(p, w) = h(p, v(p, w))$
Hicksian Demand and Expenditure Function

- Shephard’s lemma $h(p, u) = D_p e(p, u)$
- Furthermore:
  - $D_p h(p, u) = D_p^2 e(p, u)$
  - $D_p h(p, u)$ is a symmetric and NSD matrix.
  - $D_p h(p, u)p = 0$
- Notes, NSD of $D_p h$ follows from concavity of $e(p, u)$ in prices.
- Last property is a result of HD0 of $h(p, u)$ in prices.
- Symmetry is linked to transitivity (or no cycle in preferences)
Complements and Substitutes

- Goods $\ell$ and $k$ are
  - complements iff $\frac{\partial h_\ell(p, u)}{\partial p_k} \leq 0$
    - an increase in $p_k$ reduces compensated demand for goods $\ell$
      i.e. they are consumed together!
  - substitutes iff $\frac{\partial h_\ell(p, u)}{\partial p_k} \geq 0$
    - an increase in $p_k$ increases compensated demand for good $\ell$
      i.e. one good is replacing the other!

- Notes:
  - These definitions are local at $(p, u)$.
  - Symmetry of $D_p h(p, u)$ is important for this definition to make any sense!
  - Since $D_p h(p, u)p = 0$ and $\frac{\partial h_\ell}{\partial p_\ell} \leq 0$
    $\exists k \neq \ell$ s.th. $\frac{\partial h_\ell}{\partial p_k} \geq 0$, each good must have at least one substitute!
  - Using $x(p, w)$ we can define *gross* complements and substitutes.
Example

- Cobb-Douglas utility:
  \[ h(p, u) = \left( \left( \frac{\alpha p_2}{(1-\alpha)p_1} \right)^{1-\alpha}, \left( \frac{(1-\alpha)p_1}{\alpha p_2} \right)^\alpha \right) u \]

- Observe that

\[ D_p h(p, u) = \begin{pmatrix} -(1-\alpha) \frac{h_1}{p_1} & (1-\alpha) \frac{h_1}{p_2} \\ \alpha \frac{h_2}{p_1} & -\alpha \frac{h_2}{p_2} \end{pmatrix} \]

- Confirm \( \alpha \frac{h_2}{p_1} = (1-\alpha) \frac{h_1}{p_2} \).

- Good 1 and 2 are substitutes because \( \alpha \frac{h_2}{p_1} > 0 \)
Example (cont’d)

- Notice $x(p, w) = \left( \alpha \frac{w}{p_1}, (1 - \alpha) \frac{w}{p_2} \right)$ which gives

$$D_p x(p, w) = \begin{pmatrix} -\frac{x_1}{p_1} & 0 \\ 0 & -\frac{x_2}{p_2} \end{pmatrix}$$

- Therefore goods 1 and 2 are both *gross* substitutes and complements.
Slutsky Equation

- For all \((p, w)\) and \(u = v(p, w)\) we have

\[
\frac{\partial x_\ell(p, w)}{\partial p_k} = \frac{\partial h_\ell(p, u)}{\partial p_k} - \frac{\partial x_\ell(p, w)}{\partial w} x_k(p, w)
\]

Or in matrix form

\[
D_p x(p, w) = D_p h(p, u) - D_w x(p, w)x(p, w)^T
\]

- Notes:
  1. This describes the relationship between slope of compensated and ordinary demand function at a given point.
  2. For normal goods: \(\frac{\partial h_\ell(p, u)}{\partial p_\ell} \geq \frac{\partial x_\ell(p, w)}{\partial p_\ell}\)
  3. For inferior goods: \(\frac{\partial h_\ell(p, u)}{\partial p_\ell} < \frac{\partial x_\ell(p, w)}{\partial p_\ell}\)

- Graphical representation?
Hicks Decomposition of a Demand Change

- The Slutsky equation decomposes a demand change induced by a price change into two separate effects:
  - substitution effect
  - income effect
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Welfare Evaluation

- What is welfare analysis?
  - positive vs. normative perspectives.
- Consider a change in prices from $p^0$ to $p^1$.
  - Is the consumer better off?
- Utility function and the corresponding indirect utility give level of well-being as a function of prices and wealth.
  - better off iff $v(p^1, w) - v(p^0, w) > 0$.
- How do we get a quantifiable measure of the magnitude of welfare change?
  - Ordinality of utility
Money Metric Utility Function

- Given \( p \) and \( x \), how much money (wealth) is required to attain \( u(x) \)?

\[
m(p, x) \equiv e(p, u(x)) = \min_{z \geq 0} p \cdot z
\]

\[
\text{s.t. } u(z) \geq u(x)
\]

- Holding \( p \) constant, \( m(p, x) \) is in fact a utility function. Why?
Money Metric Indirect Utility Function

- Money metric indirect utility function is defined as

\[ \mu(p; q, w) \equiv e(p, v(q, w)) \]

- What does this measure?
- Fixing \( q \) and \( w \), it behaves like an expenditure function w.r.t. \( p \). Why?
- Fixing \( p \), it behaves like an indirect utility function w.r.t. \( q \). and \( w \). Why?
Quantifying welfare changes

- We use the money metric utility function to quantify $v(p^1, w) - v(p^0, w)$
  - What is the difference between expenditure needed to buy $v(p^1, w)$ and $v(p^0, w)$?
  - If we want to buy these utilities at $\bar{p}$ then

  $$\mu(\bar{p}; p^1, w) - \mu(\bar{p}; p^0, w) = e(\bar{p}, v(p^1, w)) - e(\bar{p}, v(p^0, w))$$
How to Choose $\bar{p}$?

- You are free to choose any $\bar{p}$.
- Two choices are more popular.
- Equivalent variation (set $\bar{p} = p^0$)

$$EV(p^0, p^1, w) = e(p^0, v(p^1, w)) - e(p^0, v(p^0, w)) = e(p^0, u^1) - w$$

- If the prices were fixed at $p^0$, by how much should we change wealth to reach the post change utility level ($u^1$)?

- Compensating variation (set $\bar{p} = p^1$)

$$CV(p^0, p^1, w) = e(p^1, v(p^1, w)) - e(p^1, v(p^0, w)) = w - e(p^1, u^0)$$

- If the prices were fixed at $p^1$, by how much should we change wealth to reach pre change utility level ($u^0$)?
EV vs. CV

- $p^0$: initial price $\rightarrow p^1$: final price
- $u^0$: initial level of utility $\rightarrow u^1$: final level of utility
- $v(p, w)$: indirect utility function
- EV: Starting from initial prices, how much extra wealth does the consumer need to get the final level of utility?
  \[ v(p^0, w + EV) = u^1 \]
  - What is the equivalent wealth increase that makes the consumer indifferent with the price change?
- CV: Starting from final prices, how much less wealth does the consumer need to get the initial level of utility?
  \[ v(p^1, w - CV) = u^0 \]
  - After a price change, what is the compensation to leave the consumer indifferent?
Graphical Representation of EV

\[ EV = e(p^0, u^1) - w \]
Graphical Representation of CV

$\mathbf{CV} = w - e(p^1, u^0)$

Points:
- $x(p^0, w)$
- $x(p^1, w)$
- $x(p^1, e(p^1, u^0))$

Lines:
- $u^0$
- $u^1$

Axes:
- $x_1$
- $x_2$
EV and CV as Areas under $h(p, u)$

- Assume only $p_1^0 \to p_1^1$, given the fact that $\frac{\partial e}{\partial p_1} = h_1$ we can write

$$EV = e(p^0, u^1) - w$$

$$= e(p^0, u^1) - e(p^1, u^1) = \int_{p_1^0}^{p_1^1} h_1(p_1, \overline{p}_{-1}, u^1) dp_1$$

- Similarly for CV

$$CV = w - e(p^1, u^0)$$

$$= e(p^0, u^0) - e(p^1, u^0) = \int_{p_1^0}^{p_1^1} h_1(p_1, \overline{p}_{-1}, u^0) dp_1$$

- Only difference between $EV$ and $CV$ is the utility level that $h_1$ is calculated at (inside the integral).
- When is $EV > CV$? When is $EV < CV$?
- Could you find a situation that $EV = CV$?
Deadweight Loss (DWL) of Taxation

- Consider a per unit tax \( p_1^1 = p_0^0 + t \), fix \( p_2^1 = p_2^0 \).
- Total revenue raised from this tax is \( T = tx_1(p_1^1, w) \).
- Is the consumer better off if we replace the per unit tax with a lump-sum tax equal to \( T \)?

\[
EV = e(p_0^0, u^1) - w < -T
\]

\[
e(p_0^0, u^1) < w - T
\]

- RHS: net wealth under lump-sum tax
  LHS: net wealth that delivers post-tax utility at pre-tax prices

- DWL is

\[
-T - [e(p_0^0, u^1) - w] = -th_1(p_1^1, p_2^0, u^1) - \int_{p_1^1}^{p_0^0} h_1(p_1, p_2^0, u^1)dp_1 = \int_{p_1^0}^{p_1^1} \left[ h_1(p_1, p_2^0, u^1) - h_1(p_1^1, p_2^0, u^1) \right] dp_1
\]
Consumer Surplus (CS)

- What if we don’t know $h(p, u)$? How should we measure changes in welfare?
- One way is to calculate the areas from Walrasian demand functions
- Consumer surplus is

$$CS = \int_{p_1}^{p_0} x_1(p_1, \bar{p}_{-1}, w) dp_1$$

- sometime the term Area Variation is used instead.
- What is the relation between CV, EV, and CS?
- Normal good

$$EV > CS > CV$$

- Inferior good

$$EV < CS < CV$$

- However, if several prices change the rankings might not be obvious.
Summary

• In this topic, we have
  ■ discussed preference relations and their properties
  ■ found conditions for existence of $u(x)$
  ■ learned how to derive $x(p, w)$ (and $v(p, w)$) from UMP
  ■ learned how to derive $h(p, u)$ (and $e(p, u)$) from EMP
  ■ discussed the link between UMP and EMP
  ■ introduced welfare measurement