Market Risk VaR: Model-Building Approach

Chapter 15
The Model-Building Approach

- The main alternative to historical simulation is to make assumptions about the probability distributions of the returns on the market variables.
- This is known as the model building approach (or sometimes the variance-covariance approach).
Microsoft Example (page 323-324)

- We have a position worth $10 million in Microsoft shares
- The volatility of Microsoft is 2% per day (about 32% per year)
- We use $N=10$ and $X=99$
Microsoft Example continued

- The standard deviation of the change in the portfolio in 1 day is $200,000
- The standard deviation of the change in 10 days is

\[ 200,000 \sqrt{10} = 632,456 \]
Microsoft Example continued

- We assume that the expected change in the value of the portfolio is zero (This is OK for short time periods)
- We assume that the change in the value of the portfolio is normally distributed
- Since $\mathcal{N}(-2.33)=0.01$, the VaR is

$$2.33 \times 632,456 = \$1,473,621$$
AT&T Example

- Consider a position of $5 million in AT&T
- The daily volatility of AT&T is 1% (approx 16% per year)
- The SD per 10 days is
  
  \[50,000\sqrt{10} = \$158,144\]

- The VaR is
  
  \[158,114 \times 2.33 = \$368,405\]
Now consider a portfolio consisting of both Microsoft and AT&T
Suppose that the correlation between the returns is 0.3
S.D. of Portfolio

• A standard result in statistics states that

\[ \sigma_{X+Y} = \sqrt{\sigma_X^2 + \sigma_Y^2 + 2\rho\sigma_X\sigma_Y} \]

• In this case \( \sigma_X = 200,000 \) and \( \sigma_Y = 50,000 \) and \( \rho = 0.3 \). The standard deviation of the change in the portfolio value in one day is therefore 220,227
VaR for Portfolio

- The 10-day 99% VaR for the portfolio is 
  \[ 220,227 \times \sqrt{10} \times 2.33 = $1,622,657 \]
- The benefits of diversification are 
  \[ (1,473,621 + 368,405) - 1,622,657 = $219,369 \]
- What is the incremental effect of the AT&T holding on VaR?
The Linear Model

We assume

- The daily change in the value of a portfolio is linearly related to the daily returns from market variables
- The returns from the market variables are normally distributed
Variance of change in Portfolio Value

\[ \Delta P = \sum_{i=1}^{n} \alpha_i \Delta x_i \]

\[ \sigma_P^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij} \alpha_i \alpha_j \sigma_i \sigma_j \]

\[ \sigma_P^2 = \sum_{i=1}^{n} \alpha_i^2 \sigma_i^2 + 2 \sum_{i<j} \rho_{ij} \alpha_i \alpha_j \sigma_i \sigma_j \]

\( \sigma_i \) is the daily volatility of the \( i \)th asset (i.e., SD of daily returns)

\( \sigma_P \) is the SD of the change in the portfolio value per day

\( \alpha_i \) is the amount invested in \( i \)th asset

\( \rho_{ij} \) is correlation between returns of \( i \)th and \( j \)th assets
Markowitz Result for Variance of Return on Portfolio

\[
\text{Variance of Portfolio Return} = \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij} w_i w_j \sigma_i \sigma_j
\]

- \(w_i\) is weight of \(i\)th asset in portfolio
- \(\sigma_i^2\) is variance of return on \(i\)th asset in portfolio
- \(\rho_{ij}\) is correlation between returns of \(i\)th and \(j\)th assets
Covariance Matrix $(\text{var}_i = \text{cov}_{ii})$
(page 328)

\[ C = \begin{pmatrix}
\text{var}_1 & \text{cov}_{12} & \text{cov}_{13} & \cdots & \text{cov}_{1n} \\
\text{cov}_{21} & \text{var}_2 & \text{cov}_{23} & \cdots & \text{cov}_{2n} \\
\text{cov}_{31} & \text{cov}_{32} & \text{var}_3 & \cdots & \text{cov}_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\text{cov}_{n1} & \text{cov}_{n2} & \text{cov}_{n3} & \cdots & \text{var}_n 
\end{pmatrix} \]
Alternative Expressions for $\sigma_P^2$

\[
\sigma_P^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \text{cov}_{ij} \alpha_i \alpha_j
\]

\[
\sigma_P^2 = \alpha^T C \alpha
\]

where $\alpha$ is the column vector whose $i$th element is $\alpha_i$ and $\alpha^T$ is its transpose
Example: Portfolio on Sept 25, 2008

<table>
<thead>
<tr>
<th>Index</th>
<th>Amount Invested ($000s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>DJIA</td>
<td>4,000</td>
</tr>
<tr>
<td>FTSE 100</td>
<td>3,000</td>
</tr>
<tr>
<td>CAC 40</td>
<td>1,000</td>
</tr>
<tr>
<td>Nikkei 225</td>
<td>2,000</td>
</tr>
<tr>
<td>Total</td>
<td>10,000</td>
</tr>
</tbody>
</table>
Table 13.3  Correlation matrix with same weight for all data: variable 1 is DJIA; variable 2 is FTSE 100; variable 3 is CAC 40; variable 4 is Nikkei 225.

\[
\begin{bmatrix}
1 & 0.534 & 0.569 & 0.124 \\
0.534 & 1 & 0.926 & 0.402 \\
0.569 & 0.926 & 1 & 0.409 \\
0.124 & 0.402 & 0.409 & 1
\end{bmatrix}
\]
### Table 13.4

Covariance matrix with same weight for all data: variable 1 is DJIA; variable 2 is FTSE 100; variable 3 is CAC 40; variable 4 is Nikkei 225.

\[
\begin{bmatrix}
0.0001227 & 0.0000760 & 0.0000845 & 0.0000202 \\
0.0000760 & 0.0001649 & 0.0001594 & 0.0000757 \\
0.0000845 & 0.0001594 & 0.0001795 & 0.0000804 \\
0.0000202 & 0.0000757 & 0.0000804 & 0.0002156
\end{bmatrix}
\]
Table 13.5  Covariance matrix when EWMA with $\lambda = 0.94$ is used: variable 1 is DJIA; variable 2 is FTSE 100; variable 3 is CAC 40; variable 4 is Nikkei 225.

\[
\begin{bmatrix}
0.0004896 & 0.0004324 & 0.0004379 & 0.0000763 \\
0.0004324 & 0.0009006 & 0.0008801 & 0.0004202 \\
0.0004379 & 0.0008801 & 0.0008986 & 0.0003902 \\
0.0000763 & 0.0004202 & 0.0003902 & 0.0004232
\end{bmatrix}
\]
Table 13.7  Correlation matrix when EWMA method is used: variable 1 is DJIA; variable 2 is FTSE 100; variable 3 is CAC 40; variable 4 is Nikkei 225.

\[
\begin{bmatrix}
1 & 0.649 & 0.660 & 0.168 \\
0.649 & 1 & 0.975 & 0.678 \\
0.660 & 0.975 & 1 & 0.633 \\
0.168 & 0.678 & 0.633 & 1
\end{bmatrix}
\]
### Four Index Example Using Last 500 Days of Data to Estimate Covariances

<table>
<thead>
<tr>
<th></th>
<th>Equal Weight</th>
<th>EWMA: $\lambda=0.94$</th>
</tr>
</thead>
<tbody>
<tr>
<td>One-day 99% VaR</td>
<td>$217,757</td>
<td>$471,025</td>
</tr>
</tbody>
</table>
Volatilities and Correlations
Increased in Sept 2008

Volatilities (% per day)

<table>
<thead>
<tr>
<th></th>
<th>DJIA</th>
<th>FTSE</th>
<th>CAC</th>
<th>Nikkei</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equal Weights</td>
<td>1.11</td>
<td>1.42</td>
<td>1.40</td>
<td>1.38</td>
</tr>
<tr>
<td>EWMA</td>
<td>2.19</td>
<td>3.21</td>
<td>3.09</td>
<td>1.59</td>
</tr>
</tbody>
</table>

Correlations

\[
\begin{pmatrix}
1 & 0.489 & 0.496 & -0.062 \\
0.489 & 1 & 0.918 & 0.201 \\
0.496 & 0.918 & 1 & 0.211 \\
-0.062 & 0.201 & 0.211 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0.611 & 0.629 & -0.113 \\
0.611 & 1 & 0.971 & 0.409 \\
0.629 & 0.971 & 1 & 0.342 \\
-0.113 & 0.409 & 0.342 & 1
\end{pmatrix}
\]

Equal weights

EWMAs
Alternatives for Handling Interest Rates

- Duration approach: Linear relation between $\Delta P$ and $\Delta y$, but assumes only parallel shifts
- Cash flow mapping: Variables are zero-coupon bond prices with about 10 different maturities
- Principal components analysis: 2 or 3 independent shifts with their own volatilities
Handling Interest Rates: Cash Flow Mapping

- We choose as market variables zero-coupon bond prices with standard maturities (1mm, 3mm, 6mm, 1yr, 2yr, 5yr, 7yr, 10yr, 30yr)
- Suppose that the 5yr rate is 6% and the 7yr rate is 7% and we will receive a cash flow of $10,000 in 6.5 years.
- The volatilities per day of the 5yr and 7yr bonds are 0.50% and 0.58% respectively
Example continued

- We interpolate between the 5yr rate of 6% and the 7yr rate of 7% to get a 6.5yr rate of 6.75%
- The PV of the $10,000 cash flow is

\[
\frac{10,000}{1.0675^{6.5}} = 6,540
\]
Example continued

- We interpolate between the 0.5% volatility for the 5yr bond price and the 0.58% volatility for the 7yr bond price to get 0.56% as the volatility for the 6.5yr bond.
- We allocate $\alpha$ of the PV to the 5yr bond and $(1- \alpha)$ of the PV to the 7yr bond.
Example continued

- Suppose that the correlation between movement in the 5yr and 7yr bond prices is 0.6
- To match variances

\[
0.56^2 = 0.5^2 \alpha^2 + 0.58^2 (1-\alpha)^2 + 2 \times 0.6 \times 0.5 \times 0.58 \times \alpha (1-\alpha)
\]

- This gives \( \alpha = 0.074 \)
Example continued

The value of 6,540 received in 6.5 years is replaced by
\[ 6,540 \times 0.074 = \$484 \]
in 5 years and by
\[ 6,540 \times 0.926 = \$6,056 \]
in 7 years.

This cash flow mapping preserves value and variance.
Using a PCA to Calculate VaR

- Suppose we calculate
  \[ \Delta P = -0.05 f_1 - 3.87 f_2 \]
  where \( f_1 \) is the first factor and \( f_2 \) is the second factor

- If the SD of the factor scores are 17.55 and 4.77 the SD of \( \Delta P \) is
  \[ \sqrt{0.05^2 \times 17.55^2 + 3.87^2 \times 4.77^2} = 18.48 \]
When Linear Model Can be Used

- Portfolio of stocks
- Portfolio of bonds
- Forward contract on foreign currency
- Interest-rate swap
But the Distribution of the Daily Return on an Option is not Normal

The linear model fails to capture skewness in the probability distribution of the portfolio value.
Quadratic Model
(page 338-340)

For a portfolio dependent on a single asset price it is approximately true that

\[ \Delta P = \delta \Delta S + \frac{1}{2} \gamma (\Delta S)^2 \]

so that

\[ \Delta P = S \delta \Delta x + \frac{1}{2} S^2 \gamma (\Delta x)^2 \]

Moments are

\[ E(\Delta P) = 0.5 S^2 \gamma \sigma^2 \]
\[ E(\Delta P^2) = S^2 \delta^2 \sigma^2 + 0.75 S^4 \gamma^2 \sigma^4 \]
\[ E(\Delta P^3) = 4.5 S^4 \delta^2 \gamma \sigma^4 + 1.875 S^6 \gamma^3 \sigma^6 \]
Quadratic Model continued

- When there are a small number of underlying market variable moments can be calculated analytically from the delta/gamma approximation.
- The Cornish–Fisher expansion can then be used to convert moments to fractiles.
- However when the number of market variables becomes large this is no longer feasible.
Monte Carlo Simulation

To calculate VaR using MC simulation we

- Value portfolio today
- Sample once from the multivariate distributions of the $\Delta x_i$
- Use the $\Delta x_i$ to determine market variables at end of one day
- Revalue the portfolio at the end of day
Monte Carlo Simulation continued

- Calculate $\Delta P$
- Repeat many times to build up a probability distribution for $\Delta P$
- VaR is the appropriate fractile of the distribution times square root of $N$
- For example, with 1,000 trial the 1 percentile is the 10th worst case.
Speeding up Calculations with the Partial Simulation Approach

- Use the approximate delta/gamma relationship between $\Delta P$ and the $\Delta x_i$ to calculate the change in value of the portfolio

$$
\Delta P = \sum_{i=1}^{n} S_i \delta_i \Delta x_i + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{2} S_i S_j \gamma_{i,j} \Delta x_i \Delta x_j
$$

- This is also a way of speeding up computations in the historical simulation approach
Alternative to Normal Distribution Assumption in Monte Carlo

- In a Monte Carlo simulation we can assume non-normal distributions for the $x_i$ (e.g., a multivariate t-distribution)
- Can also use a Gaussian or other copula model in conjunction with empirical distributions
Model Building vs Historical Simulation

Model building approach can be used for investment portfolios where there are no derivatives, but it does not usually work when for portfolios where
- There are derivatives
- Positions are close to delta neutral