Volatility

Chapter 9
Definition of Volatility

- The volatility $\sigma$ of a variable is the standard deviation of its return per unit of time (one year, one day) with the return being expressed with continuous compounding.
\[ \ln\left( \frac{S_T}{S_0} \right) : \text{total return earned in time } T \]

\[ \sigma \sqrt{T} : \text{the standard deviation of } \ln\left( \frac{S_T}{S_0} \right) \]

when \( T \) is small, the continuously compounded return is close to the percentage change

i.e. \[ \ln\left( \frac{S_T}{S_0} \right) \approx \frac{S_T - S_0}{S_0} \]
Definition of Volatility

● Example:

\[ S_0 = $50 \]
\[ \sigma = 30\% \text{ per year} \]

std. of percentage change in stock price in one week approximately: \[ 30 \times \sqrt{1/52} = 4.16\% \]

A one–standard–deviation move in the stock price in one week: \[ 50 \times 0.0416 = $2.08 \]
Some points concerning volatility

- The variance rate is the square of volatility
- Standard deviation of returns increase with the square root of time, the variance of this return increases linearly with time
- Normally days when markets are closed are ignored in volatility calculations (252 days per year)
Implied Volatilities

- Implied volatilities are the volatilities implied from option prices
- Of the variables needed to price an option the one that cannot be observed directly is volatility
- We can therefore imply volatilities from market prices and vice versa
Estimating Volatility from Historical Data

1. Take observations \( S_0, S_1, \ldots, S_n \) at intervals of \( \tau \) years

2. Calculate the continuously compounded return in each interval as:

\[
u_i = \ln \left( \frac{S_i}{S_{i-1}} \right)\]

3. Calculate the standard deviation, \( s \), of the \( u_i \)'s

4. The historical volatility estimate is:

\[
\hat{\sigma} = \frac{s}{\sqrt{\tau}}
\]
Are Daily Percentage Changes in Financial Variables Normal?

- The Black-Scholes-Merton Model assumes that asset prices change continuously and have constant volatility. This means that the return in a short period of time \( \Delta t \) always has a normal distribution with a standard deviation of \( \sigma \sqrt{\Delta t} \).
Daily movements of 12 different exchange rates over ten years

<table>
<thead>
<tr>
<th></th>
<th>Real World (%)</th>
<th>Normal Model (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>&gt;1 SD</td>
<td>25.04</td>
<td>31.73</td>
</tr>
<tr>
<td>&gt;2SD</td>
<td>5.27</td>
<td>4.55</td>
</tr>
<tr>
<td>&gt;3SD</td>
<td>1.34</td>
<td>0.27</td>
</tr>
<tr>
<td>&gt;4SD</td>
<td>0.29</td>
<td>0.01</td>
</tr>
<tr>
<td>&gt;5SD</td>
<td>0.08</td>
<td>0.00</td>
</tr>
<tr>
<td>&gt;6SD</td>
<td>0.03</td>
<td>0.00</td>
</tr>
</tbody>
</table>

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An Alternative to Normal Distributions: The Power Law

The value $v$ of the variable satisfies

$$\text{Prob}(v > x) = Kx^{-\alpha}$$

For $x$ large, where $K$ and $\alpha$ are constants

This seems to fit the behavior of the returns on many market variables better than the normal distribution
The Power Law

- Example:

Suppose a power law with $\alpha = 3$

Observation: $\Pr(v > 10) = 0.05$

Then: $K = 50$, and we can estimate

$\Pr(v > 20) = 0.00625$

$\Pr(v > 30) = 0.0019$, ...
The Power Law

We have

\[ \ln[\Pr(v > x)] = \ln K - \alpha \ln x \]

So by plotting \( \ln[\Pr(v > x)] \) against \( \ln x \),
we can test whether the power law holds.
Figure 5.2  Log-log plot for exchange rate increases: $x$ is number of standard deviations; $v$ is the exchange rate increase.
Monitoring Daily Volatility

- Define $\sigma_n$ as the volatility per day between day $n-1$ and day $n$, as estimated at end of day $n-1$.
- Define $S_i$ as the value of market variable at end of day $i$.
- Define $u_i = \ln(S_i/S_{i-1})$.

\[ \sigma_n^2 = \frac{1}{m-1} \sum_{i=1}^{m} (u_{n-i} - \bar{u})^2 \]

\[ \bar{u} = \frac{1}{m} \sum_{i=1}^{m} u_{n-i} \]
Simplifications Usually Made in Risk Management

- Define $u_i$ as $(S_i - S_{i-1})/S_{i-1}$
- Assume that the mean value of $u_i$ is zero
- Replace $m-1$ by $m$

This gives

$$\sigma_n^2 = \frac{1}{m} \sum_{i=1}^{m} u_{n-i}^2$$
Weighting Scheme

Instead of assigning equal weights to the observations we can set

\[ \sigma_n^2 = \sum_{i=1}^{m} \alpha_i u_{n-i}^2 \]

where

\[ \sum_{i=1}^{m} \alpha_i = 1 \]
ARCH(m) Model

In an ARCH(m) model we also assign some weight to the long-run variance rate, $V_L$:

$$\sigma_n^2 = \gamma V_L + \sum_{i=1}^{m} \alpha_i u_{n-i}^2$$

where

$$\gamma + \sum_{i=1}^{m} \alpha_i = 1$$
EWMA Model

• In an exponentially weighted moving average (EWMA) model, the weights assigned to the $u^2$ decline exponentially as we move back through time, i.e. $\alpha_{i+1} = \lambda \alpha_i$
  where $\lambda$ is between 0 and 1

• This leads to

$$\sigma_n^2 = \lambda \sigma_{n-1}^2 + (1 - \lambda)u_{n-1}^2$$
EWMA Model

Substitute for $\sigma^2_n$ in

$$\sigma^2_n = \lambda \sigma^2_{n-1} + (1 - \lambda)u^2_{n-1}$$

to get

$$\sigma^2_n = (1 - \lambda)(u^2_{n-1} + \lambda u^2_{n-2}) + \lambda^2 \sigma^2_{n-2}$$

continuing in this way, we see that

$$\sigma^2_n = (1 - \lambda) \sum_{i=1}^{m} \lambda^{i-1} u^2_{n-i} + \lambda^m \sigma^2_{n-m}$$

for large $m$, this equation is the same as

$$\sigma^2_n = \sum_{i=1}^{m} \alpha_i u^2_{n-i} \text{ for } \alpha_i = (1 - \lambda) \lambda^{i-1}$$
Attractions of EWMA

- Relatively little data needs to be stored
- We need only remember the current estimate of the variance rate and the most recent observation on the market variable
- Tracks volatility changes
- RiskMetrics uses $\lambda = 0.94$ for daily volatility forecasting
The GARCH (1,1) Model

In GARCH (1,1) we assign some weight to the long-run average variance rate

\[ \sigma_n^2 = \gamma V_L + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2 \]

Since weights must sum to 1

\[ \gamma + \alpha + \beta = 1 \]
GARCH (1,1) continued

Setting $\omega = \gamma V$ the GARCH (1,1) model reads as

$$\sigma_n^2 = \omega + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2$$

and

$$V_L = \frac{\omega}{1 - \alpha - \beta} = \frac{\omega}{\gamma}$$
Example

- Suppose
  \[ \sigma_n^2 = 0.000002 + 0.13u_{n-1}^2 + 0.86\sigma_{n-1}^2 \]
  \[ \gamma = 1 - \alpha - \beta = 1 - 0.13 - 0.86 = 0.01 \]
  \[ \omega = \gamma V_L \Rightarrow V_L = 0.0002 \]

- The long-run variance rate is 0.0002 so that the long-run volatility per day is 1.4%
Example continued

- Suppose that the current estimate of the volatility is 1.6% per day and the most recent percentage change in the market variable is 1%.
- The new variance rate is
  \[0.000002 + 0.13 \times 0.0001 + 0.86 \times 0.000256 = 0.00023336\]
The new volatility is 1.53% per day
In maximum likelihood methods we choose parameters that maximize the likelihood of the observations occurring.
Example 1

- We observe that a certain event happens one time in ten trials. What is our estimate of the proportion of the time, $p$, that it happens?
- The probability of the outcome is
  \[ p(1-p)^9 \]
- We maximize this to obtain a maximum likelihood estimate: $p=0.1$
Example 2

Estimate the variance of observations from a normal distribution with mean zero

Maximize: \[
\prod_{i=1}^{n} \left[ \frac{1}{\sqrt{2\pi\nu}} \exp \left( \frac{-u_i^2}{2\nu} \right) \right]
\]
or:
\[
\sum_{i=1}^{n} \left[ -\ln(\nu) - \frac{u_i^2}{\nu} \right]
\]

This gives: \[
\nu = \frac{1}{n} \sum_{i=1}^{n} u_i^2
\]
Application to GARCH (1,1)

We choose parameters that maximize

\[ \sum_{i=1}^{n} \left[ -\ln(v_i) - \frac{u_i^2}{v_i} \right] \]
Application to GARCH (1,1), Example

- The next table analyzes data on the Japanese yen exchange rate between January 6, 1988, and August 15, 1997. The numbers are based on trial estimates of the three parameters: $\omega$, $\alpha$, $\beta$. 
### Table 5.3: Estimation of parameters in GARCH(1,1) model.

<table>
<thead>
<tr>
<th>Date</th>
<th>Day</th>
<th>$S_i$</th>
<th>$u_i$</th>
<th>$v_i = \sigma_i^2$</th>
<th>$-\ln(v_i) - \frac{u_i^2}{v_i}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>06-Jan-88</td>
<td>1</td>
<td>0.007728</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>07-Jan-88</td>
<td>2</td>
<td>0.007779</td>
<td>0.006599</td>
<td></td>
<td></td>
</tr>
<tr>
<td>08-Jan-88</td>
<td>3</td>
<td>0.007746</td>
<td>-0.004242</td>
<td>0.000004355</td>
<td>9.6283</td>
</tr>
<tr>
<td>11-Jan-88</td>
<td>4</td>
<td>0.007816</td>
<td>0.009037</td>
<td>0.000004198</td>
<td>8.1329</td>
</tr>
<tr>
<td>12-Jan-88</td>
<td>5</td>
<td>0.007837</td>
<td>0.002687</td>
<td>0.000004455</td>
<td>9.8568</td>
</tr>
<tr>
<td>13-Jan-88</td>
<td>6</td>
<td>0.007924</td>
<td>0.011101</td>
<td>0.000004220</td>
<td>7.1529</td>
</tr>
<tr>
<td>13-Aug-97</td>
<td>2421</td>
<td>0.008643</td>
<td>0.003374</td>
<td>0.000007626</td>
<td>9.3321</td>
</tr>
<tr>
<td>14-Aug-97</td>
<td>2422</td>
<td>0.008493</td>
<td>-0.017309</td>
<td>0.000007092</td>
<td>5.3294</td>
</tr>
<tr>
<td>15-Aug-97</td>
<td>2423</td>
<td>0.008495</td>
<td>0.000144</td>
<td>0.000008417</td>
<td>9.3824</td>
</tr>
</tbody>
</table>

**Trial estimates of GARCH parameters**

<table>
<thead>
<tr>
<th>$\omega$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00000176</td>
<td>0.0626</td>
<td>0.8976</td>
</tr>
</tbody>
</table>
Figure 5.3  Daily volatility of the yen/USD exchange rate, 1988-97.
Variance Targeting

- One way of implementing GARCH(1,1) that increases stability is by using variance targeting
- We set the long-run average volatility equal to the sample variance
- Only two other parameters then have to be estimated
How Good is the Model?

The assumption underlying a GARCH model is that volatility changes with the passage of time, during some periods high, during others relatively low. In other words, when $u_i^2$ is high, there is a tendency for $u_{i+1}^2, u_{i+2}^2, ...$ to be high and vice versa.
How Good is the Model?

- So assuming that the $u_i^2$ exhibit autocorrelation. If a GARCH model is working well, the autocorrelation for the variables $u_i^2 / \sigma_i^2$ should be very small.
Table 5.4 Autocorrelations before and after the use of a GARCH model.

<table>
<thead>
<tr>
<th>Time lag</th>
<th>Autocorrelation for $u_i^2$</th>
<th>Autocorrelation for $u_i^2/\sigma_i^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.072</td>
<td>0.004</td>
</tr>
<tr>
<td>2</td>
<td>0.041</td>
<td>-0.005</td>
</tr>
<tr>
<td>3</td>
<td>0.057</td>
<td>0.008</td>
</tr>
<tr>
<td>4</td>
<td>0.107</td>
<td>0.003</td>
</tr>
<tr>
<td>5</td>
<td>0.075</td>
<td>0.016</td>
</tr>
<tr>
<td>6</td>
<td>0.066</td>
<td>0.008</td>
</tr>
<tr>
<td>7</td>
<td>0.019</td>
<td>-0.033</td>
</tr>
<tr>
<td>8</td>
<td>0.085</td>
<td>0.012</td>
</tr>
<tr>
<td>9</td>
<td>0.054</td>
<td>0.010</td>
</tr>
<tr>
<td>10</td>
<td>0.030</td>
<td>-0.023</td>
</tr>
<tr>
<td>11</td>
<td>0.038</td>
<td>-0.004</td>
</tr>
<tr>
<td>12</td>
<td>0.038</td>
<td>-0.021</td>
</tr>
<tr>
<td>13</td>
<td>0.057</td>
<td>-0.001</td>
</tr>
<tr>
<td>14</td>
<td>0.040</td>
<td>0.002</td>
</tr>
<tr>
<td>15</td>
<td>0.007</td>
<td>-0.028</td>
</tr>
</tbody>
</table>
How Good is the Model?

data. For a more scientific test, we can use what is known as the Ljung-Box statistic.\textsuperscript{13} If a certain series has $m$ observations the Ljung-Box statistic is

$$m \sum_{k=1}^{K} \hat{w}_k \hat{\eta}_k^2$$

where $\eta_k$ is the autocorrelation for a lag of $k$, $K$ is the number of lags considered, and

$$w_k = \frac{m + 2}{m - k}$$

For $K = 15$, zero autocorrelation can be rejected with 95% confidence when the Ljung-Box statistic is greater than 25.
Forecasting Future Volatility

A few lines of algebra shows that

\[ E[\sigma_{n+k}^2] = V_L + (\alpha + \beta)^k (\sigma_n^2 - V_L) \]

\( \alpha + \beta < 1 \): mean reverting

\( \alpha + \beta > 1 \): mean fleeing
Volatility Term Structures

Define

\[ V(t) = E[\sigma^2_{n+t}], \quad a = \ln \frac{1}{\alpha + \beta} \quad \text{then} \]

\[ V(t) = V_L + e^{-at} [V(0) - V_L] \]

The average variance rate per day between today and \( T \)

\[ \frac{1}{T} \int_0^T V(t) dt = V_L + \frac{1 - e^{-aT}}{aT} [V(0) - V_L] \]
Forecasting Future Volatility

The volatility per year for an option lasting $T$ days is

$$\sigma(T) = \sqrt{252 \left\{ V_L + \frac{1 - e^{-aT}}{aT} [V(0) - V_L] \right\}}$$
Volatility Term Structures

- The GARCH (1,1) model allows us to predict volatility term structures changes.
- When $\sigma(0)$ changes by $\Delta \sigma(0)$, GARCH (1,1) predicts that $\sigma(T)$ changes by

$$
1 - e^{-aT} \frac{\sigma(0)}{\sigma(T)} \Delta \sigma(0), \text{ because}
$$

$$
\sigma(T)^2 = 252\{V_L + \frac{1 - e^{-aT}}{aT} \left( \frac{\sigma(0)^2}{252} - V_L \right) \}
$$