The VaR Measure
The Question Being Asked in VaR

“What loss level is such that we are $X\%$ confident it will not be exceeded in $N$ business days?”
VaR and Regulatory Capital

- Regulators base the capital they require banks to keep on VaR.
- The market-risk capital is $k$ times the 10-day 99% VaR where $k$ is at least 3.0.
- Under Basel II capital for credit risk and operational risk is based on a one-year 99.9% VaR.
Advantages of VaR

- It captures an important aspect of risk in a single number
- It is easy to understand
- It asks the simple question: “How bad can things get?”
VaR vs. Expected Shortfall

- VaR is the loss level that will not be exceeded with a specified probability.
- Expected shortfall is the expected loss given that the loss is greater than the VaR level (also called C-VaR and Tail Loss).
- Two portfolios with the same VaR can have very different expected shortfalls.
Distributions with the Same VaR but Different Expected Shortfalls
Coherent Risk Measures

- Define a coherent risk measure as the amount of cash that has to be added to a portfolio to make its risk acceptable.

- Properties of coherent risk measure:
  - If one portfolio always produces a worse outcome than another, its risk measure should be greater.
  - If we add an amount of cash $K$ to a portfolio, its risk measure should go down by $K$.
  - Changing the size of a portfolio by $\lambda$ should result in the risk measure being multiplied by $\lambda$.
  - The risk measures for two portfolios after they have been merged should be no greater than the sum of their risk measures before they were merged.
VaR vs Expected Shortfall

- VaR satisfies the first three conditions but not the fourth one
- Expected shortfall satisfies all four conditions.
Example. Suppose two independent projects with a 0.02 probability of a loss of $10 million and a probability of 0.98 of a loss of $1 million during one year. The one year 97.5% VaR for each is $1 million. Put in the same portfolio there is a 0.0004 probability of $20 million loss, a 0.0392 probability of $11 million loss, and a 0.9604 probability of $2 million loss. The one year 97.5% VaR for the portfolio is $11 million greater than the sum of the VaRs of the projects.
Spectral Risk Measures

- A spectral risk measure assigns weights to quantiles of the loss distribution.
- VaR assigns all weight to $X$th quantile of the loss distribution.
- Expected shortfall assigns equal weight to all quantiles greater than the $X$th quantile.
- For a coherent risk measure weights must be a non-decreasing function of the quantiles.
Normal Distribution Assumption

- The simplest assumption is that daily gains/losses are normally distributed, zero mean and independent.
- It is then easy to calculate VaR from the standard deviation (1-day VaR = $N^{-1}(X)\sigma$).
- The $N$-day VaR equals $\sqrt{N}$ times the one-day VaR.
- Regulators allow banks to calculate the 10 day VaR as $\sqrt{10}$ times the one-day VaR.
Independence Assumption in VaR Calculations

- When daily changes in a portfolio are identically normally distributed and independent with mean zero the variance over $N$ days is $N$ times the variance over one day.

- When there is autocorrelation equal to $\rho$ the multiplier is increased from $N$ to

$$N + 2(N - 1)\rho + 2(N - 2)\rho^2 + 2(N - 3)\rho^3 + \ldots 2\rho^{N-1}$$
Impact of Autocorrelation: Ratio of N-day VaR to 1-day VaR

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Choice of VaR Parameters

- Time horizon should depend on how quickly portfolio can be unwound. Regulators in effect use 1-day for bank market risk and 1-year for credit/operational risk. Fund managers often use one month.

- Confidence level depends on objectives. Regulators use 99% for market risk and 99.9% for credit/operational risk. A bank wanting to maintain a AA credit rating will often use 99.97% for internal calculations. (VaR for high confidence level cannot be observed directly from data and must be inferred in some way.)
VaR Measures for a Portfolio where an amount $x_i$ is invested in the $i$th component of the portfolio

- **Marginal VaR:** \[ \frac{\partial \text{VaR}}{\partial x_i} \]

- **Incremental VaR:** Incremental effect of $i$th component on VaR, what is the difference between VaR with and without the subportfolio?

- **Approximate formula (Component VaR):** \[ x_i \frac{\partial \text{VaR}}{\partial x_i} \]
Properties of Component VaR

- The total VaR is the sum of the component VaR (Euler’s theorem)
  \[ \text{VaR} = \sum_{i=1}^{N} \frac{\partial \text{VaR}}{\partial x_i} x_i \]

- The component VaR therefore provides a sensible way of allocating VaR to different activities
Backtesting

- Backtesting a VaR calculation methodology involves looking at how often exceptions (loss > VaR) occur in past.
- One issue in backtesting a one day VaR is whether we take account of changes made in the portfolio during a day.
- Backtesting a one-day VaR: a) compare VaR with actual change in portfolio value during the day and b) compare VaR with change in portfolio value assuming no change in portfolio composition during the day.
Backtesting

- Suppose that the theoretical probability of an exception is $p = 1 - \frac{X}{100}$.
- We look at $n$ days and observe that VaR is exceeded on $m$ days, $m/n > p$.
- Should we reject the model for calculating VaR?
Backtesting

- We consider two alternative hypotheses:
  1. The probability of an exception on any given day is $p$.
  2. The probability of an exception on any given day is greater than $p$. 
Backtesting

- The probability of $m$ or more exceptions in $n$ days is
  \[
  \sum_{k=m}^{n} \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}
  \]
- An often used confidence level in statistical test is 5%
- If this probability is less than 5%, we reject the first hypothesis that the probability of an exception is $p$. 
Backtesting, Example

- Backtesting a VaR(99%) using 600 days. We observe 9 exceptions, the expected number is 6. Using the previous formula, the probability of nine or more exceptions is 0.152, so, at a 5% confidence level we should not reject the model. If the exceptions had been 12, the probability would be 0.019 and the model rejected.
Backtesting

- When the number of exceptions is lower than expected, \( m \), we can similarly compare

\[
\sum_{k=0}^{m} \frac{n!}{k!(n-k)!} p^k (1 - p)^{n-k}
\]

with the 5% threshold.
Bunching

- Bunching occurs when exceptions are not evenly spread throughout the backtesting period
- Statistical tests for bunching have been developed (See page 171)
Problem

Suppose that each of two investments has a 0.9% chance of a loss of $10 million, a 99.1% of a loss of $1 million, and zero probability of a gain. The investments are independent of each other. (a) What is the VaR for one of the investments when the confidence level is 99%? (b) What is the expected shortfall for one of the investments when the confidence level is 99%? (c) What is the VaR for a portfolio consisting of the two investments when the confidence level is 99%? (d) What is the expected shortfall for a portfolio consisting of the two investments when the confidence level is 99%? (e) Show that in this example VaR does not satisfy the Subadditivity condition whereas expected shortfall does.
Answer

8.5. (a) $1$ million, (b) $9.1$ million, (c) $11$ million, (d) $11.07$ million
(e) $1 + 1 < 11$ but $9.2 + 9.2 > 11.07$. 
Problems

Suppose that the change in the value of a portfolio over a 1-day time period is normal with a mean of zero and a standard deviation of $2 million. What is (a) the 1-day 97.5% VaR, (b) the 5-day 97.5% VaR, and (c) the 5-day 99% VaR?

What difference does it make to your answers to (b) and (c) of Problem 8.6 if there is first-order daily autocorrelation with correlation parameter equal to 0.16?
8.6. (a) $3.92$ million, (b) $8.77$ million, (c) $10.40$ million.

8.7. (b) becomes $9.96$ million and (c) becomes $11.82$ million.