Interest Rate Risk

Chapter 4
Measuring Interest Rates

- The compounding frequency used for an interest rate is the unit of measurement.
- The difference between quarterly and annual compounding is analogous to the difference between miles and kilometers.
Continuous Compounding
(Page 81)

- In the limit as we compound more and more frequently we obtain continuously compounded interest rates
- $100$ grows to $100e^{RT}$ when invested at a continuously compounded rate $R$ for time $T$
- $100$ received at time $T$ discounts to $100e^{-RT}$ at time zero when the continuously compounded discount rate is $R$
Conversion Formulas
(Page 82)

Define

$R_c$: continuously compounded rate

$R_m$: same rate with compounding $m$ times per year

$$R_c = m \ln \left( 1 + \frac{R_m}{m} \right)$$

$$R_m = m \left( e^{R_c/m} - 1 \right)$$
Zero Rates

A zero rate (or spot rate), for maturity $T$ is the rate of interest earned on an investment that provides a payoff only at time $T$. 
Forward Rates

The forward rate is the future zero rate implied by today’s term structure of interest rates.
Formula for Forward Rates

(Equation 4.5, page 84)

- Suppose that the zero rates for time periods $T_1$ and $T_2$ are $R_1$ and $R_2$ with both rates continuously compounded.
- The forward rate for the period between times $T_1$ and $T_2$ is

$$\frac{R_2 T_2 - R_1 T_1}{T_2 - T_1}$$
### Example (Table 4.2, page 83)

<table>
<thead>
<tr>
<th>Maturity (years)</th>
<th>Zero Rate (% cont comp)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>5.0</td>
</tr>
<tr>
<td>1.0</td>
<td>5.8</td>
</tr>
<tr>
<td>1.5</td>
<td>6.4</td>
</tr>
<tr>
<td>2.0</td>
<td>6.8</td>
</tr>
</tbody>
</table>
## Forward Rates (Table 4.3, page 84)

<table>
<thead>
<tr>
<th>Period (years)</th>
<th>Forward Rate (% cont comp)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5 to 1.0</td>
<td>6.6</td>
</tr>
<tr>
<td>1.0 to 1.5</td>
<td>7.6</td>
</tr>
<tr>
<td>1.5 to 2.0</td>
<td>8.0</td>
</tr>
</tbody>
</table>
Bond Pricing

- To calculate the cash price of a bond we discount each cash flow at the appropriate zero rate.
- In our example, the theoretical price of a two-year bond providing a 6% coupon semiannually is:

\[
3e^{-0.05\times0.5} + 3e^{-0.058\times1.0} + 3e^{-0.064\times1.5} + 103e^{-0.068\times2.0} = 98.39
\]
Bond Yield

- The bond yield is the discount rate that makes the present value of the cash flows on the bond equal to the market price of the bond.
- Suppose that the market price of the bond in our example equals its theoretical price of 98.39.
- The bond yield (continuously compounded) is given by solving

\[ 3e^{-y \times 0.5} + 3e^{-y \times 1.0} + 3e^{-y \times 1.5} + 103e^{-y \times 2.0} = 98.39 \]

to get \( y = 0.0676 \) or 6.76%.
Determining the Zero Curve: The Bootstrap Method

- We work forward to successively longer maturities.
- Suppose that the zero curve determined for zero to two years is as in our example and that the price of a 2.5-year bond paying a coupon of 8% is 102
- If $R$ is the 2.5-year rate we must have
  $$4e^{-0.05 \times 0.5} + 4e^{-0.058 \times 1.0} + 4e^{-0.064 \times 1.5} + 4e^{-0.68 \times 2.0} + 104e^{-R \times 2.5} = 102$$
- This can be solved to give $R=7.05\%$
The Zero Curve (Figure 4.1, page 87)
LIBOR Rates

- LIBOR: London Interbank Offered Rate
- LIBOR rates are 1-, 3-, 6-, and 12-month borrowing rates for banks that have AA credit ratings
- To extend the LIBOR zero curve we can
  - Create a zero curve to represent the rates at which AA-rates companies can borrow for longer periods of time
  - Create a zero curve to represent the future short term borrowing rates for AA-rated companies
- In practice we do the second
Risk-Free Rate (Page 89)

- In practice traders and risk managers assume that the LIBOR/swap zero curve is the risk-free zero curve.
- The Treasury curve is about 50 basis points below the LIBOR/swap zero curve, a true risk-free yield curve is about 10 basis points below the LIBOR/swap yield curve.
- Treasury rates are considered to be artificially low for a variety of regulatory and tax reasons.
The duration of an instrument is a measure of how long, on average, the holder has to wait before receiving cash payments.

Duration of a bond that provides cash flow $c_i$ at time $t_i$ is

$$
\sum_{i=1}^{n} t_i \left[ \frac{c_i e^{-yt_i}}{B} \right]
$$

where $B$ is its price and $y$ is its yield (continuously compounded).

In fact a weighted average of the times when payments are made.
Duration

- For a small change $\Delta y$ in the yield, it is approximately true that
  $$\Delta B = \frac{dB}{dy} \Delta y$$

- But
  $$\frac{1}{B} \frac{dB}{dy} = -D$$

- This leads to
  $$\frac{\Delta B}{B} = -D \Delta y$$
Duration

\[ \frac{\Delta B}{B} = -D\Delta y \]

- An approximate relationship between percentage changes in a bond price and changes in its yield
- Duration is a widely used measure of a portfolio’s exposure to yield curve movements
Calculation of Duration for a 3-year bond paying a coupon 10%. Bond yield=12%.
(Table 4.5, page 91)

<table>
<thead>
<tr>
<th>Time (yrs)</th>
<th>Cash Flow ($)</th>
<th>PV ($)</th>
<th>Weight</th>
<th>Time x Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>5</td>
<td>4.709</td>
<td>0.050</td>
<td>0.025</td>
</tr>
<tr>
<td>1.0</td>
<td>5</td>
<td>4.435</td>
<td>0.047</td>
<td>0.047</td>
</tr>
<tr>
<td>1.5</td>
<td>5</td>
<td>4.176</td>
<td>0.044</td>
<td>0.066</td>
</tr>
<tr>
<td>2.0</td>
<td>5</td>
<td>3.933</td>
<td>0.042</td>
<td>0.083</td>
</tr>
<tr>
<td>2.5</td>
<td>5</td>
<td>3.704</td>
<td>0.039</td>
<td>0.098</td>
</tr>
<tr>
<td>3.0</td>
<td>105</td>
<td>73.256</td>
<td>0.778</td>
<td>2.333</td>
</tr>
<tr>
<td>Total</td>
<td>130</td>
<td>94.213</td>
<td>1.000</td>
<td>2.653</td>
</tr>
</tbody>
</table>
Duration Continued

- When the yield $y$ is expressed with compounding $m$ times per year

$$\Delta B = - \frac{BD \Delta y}{1 + y/m}$$

- The variable

$$D^* = \frac{D}{1 + y/m}$$

is referred to as the “modified duration” and

$$\Delta B = -BD^* \Delta y$$
Risk M

4.22
Convexity (Page 94)

The convexity of a bond is defined as

\[ C = \frac{1}{B} \frac{\partial^2 B}{\partial y^2} = \frac{\sum_{i=1}^{n} c_i t_i^2 e^{-yt_i}}{B} \]

so that

\[ \frac{\Delta B}{B} = -D \Delta y + \frac{1}{2} C (\Delta y)^2 \]
Duration of Portfolios

- Suppose $P$ is the value of a portfolio of assets dependent on interest rates. Make a small parallel shift in the zero-coupon yield curve and observe the change $\Delta P$ in $P$. Duration is defined as

$$D = -\frac{1}{P} \frac{\Delta P}{\Delta y}$$

- The convexity is defined as

$$C = \frac{1}{P} \frac{d^2 P}{dy^2}$$
Duration of Portfolios

- Suppose the portfolio consists of $n$ assets being worth $X_1,\ldots, X_n$ with durations $D_1,\ldots, D_n$, then the duration of the portfolio would be

$$D = \sum_{i=1}^{n} \frac{X_i}{P} D_i$$

- The duration of a portfolio is the weighted average of the durations of the components of the portfolio. Similarly for convexity.
Nonparallel Yield Curve Shifts

- The equation

\[ \frac{\Delta P}{P} = -D\Delta y + \frac{1}{2}C(\Delta y)^2 \]

- Only quantifies exposure to parallel yield curve shifts.
### Table 4.6  Zero-coupon yield curve (rates continuously compounded)

<table>
<thead>
<tr>
<th>Maturity (years)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>7</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rate (%)</td>
<td>4.0</td>
<td>4.5</td>
<td>4.8</td>
<td>5.0</td>
<td>5.1</td>
<td>5.2</td>
<td>5.3</td>
</tr>
</tbody>
</table>
Starting Zero Curve (Figure 4.3, page 97)
Calculating a Partial Duration
(Figure 4.4, page 97)
The Partial Duration of the Portfolio

- The partial duration of the portfolio for the \( i \)th point on the zero curve is
  \[
  \frac{1}{P} \Delta P_i \Delta x_i
  \]

- Where \( \Delta x_i \) is the size of the small change made to the \( i \)th point on the yield curve and \( \Delta P_i \) the resultant change in the portfolio value.
## The Partial Duration of the Portfolio

**Table 4.7** Partial durations for a portfolio.

<table>
<thead>
<tr>
<th>Maturity (years)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>7</th>
<th>10</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Duration</td>
<td>2.0</td>
<td>1.6</td>
<td>0.6</td>
<td>0.2</td>
<td>-0.5</td>
<td>-1.8</td>
<td>-1.9</td>
<td>0.2</td>
</tr>
</tbody>
</table>
Combining Partial Durations to Create Rotation in the Yield Curve
(Figure 4.5, page 98)
Combining Partial Durations to Create Rotation in the Yield Curve

- Suppose the changes to 1-year, 2-year, 3-year, 4-year, 5-year, 7-year, and 10-year points are -3e, -2e, -e, 0, e, 3e, and 6e for some small e. The percentage change in the value of the portfolio is

\[-2 \times (-3e) + (-1.6) \times (-2e) + (-0.6) \times (-e) + (-0.2) \times (0) + 0.5 \times e + 1.8 \times 3e + 1.9 \times 6e = 27.1e\]
Interest Rate Deltas

- Interest rate delta for a portfolio: the sensitivity of the portfolio with respect to changes in interest rates

- Several deltas:
  1. The change in value for a one-basis-point parallel shift in the zero curve (DV01)
  2. The change in value for a one-basis-point for each point on the zero-coupon yield curve
Interest Rate Deltas

3. Divide the yield curve into a number of buckets and calculate for each bucket the impact of changing all the zero rates corresponding to the bucket by one-basis-point while keeping all other zero rates unchanged (GAP management)
Change When One Bucket Is Shifted  (Figure 4.6, page 99)
Principal Components Analysis

- Attempts to identify standard shifts (or factors) for the yield curve so that most of the movements that are observed in practice are combinations of the standard shifts.
Principal Component Analysis, Example

- The market variables are ten US Treasury rates with maturities between 3 months and 30 years, using 1543 daily observations between 1989 and 1995.
<table>
<thead>
<tr>
<th></th>
<th>PC1</th>
<th>PC2</th>
<th>PC3</th>
<th>PC4</th>
<th>PC5</th>
<th>PC6</th>
<th>PC7</th>
<th>PC8</th>
<th>PC9</th>
<th>PC10</th>
</tr>
</thead>
<tbody>
<tr>
<td>3m</td>
<td>0.21</td>
<td>-0.57</td>
<td>0.50</td>
<td>0.47</td>
<td>-0.39</td>
<td>-0.02</td>
<td>0.01</td>
<td>0.00</td>
<td>0.01</td>
<td>0.00</td>
</tr>
<tr>
<td>6m</td>
<td>0.26</td>
<td>-0.49</td>
<td>0.23</td>
<td>-0.37</td>
<td>0.70</td>
<td>0.01</td>
<td>-0.04</td>
<td>-0.02</td>
<td>-0.01</td>
<td>0.00</td>
</tr>
<tr>
<td>12m</td>
<td>0.32</td>
<td>-0.32</td>
<td>-0.37</td>
<td>-0.58</td>
<td>-0.52</td>
<td>-0.23</td>
<td>-0.04</td>
<td>-0.05</td>
<td>0.00</td>
<td>0.01</td>
</tr>
<tr>
<td>2y</td>
<td>0.35</td>
<td>-0.10</td>
<td>-0.38</td>
<td>0.17</td>
<td>0.04</td>
<td>0.59</td>
<td>0.56</td>
<td>0.12</td>
<td>-0.12</td>
<td>-0.05</td>
</tr>
<tr>
<td>3y</td>
<td>0.36</td>
<td>0.02</td>
<td>-0.30</td>
<td>0.27</td>
<td>0.07</td>
<td>0.24</td>
<td>-0.79</td>
<td>0.00</td>
<td>-0.09</td>
<td>-0.00</td>
</tr>
<tr>
<td>4y</td>
<td>0.36</td>
<td>0.14</td>
<td>-0.12</td>
<td>0.25</td>
<td>0.16</td>
<td>-0.63</td>
<td>0.15</td>
<td>0.55</td>
<td>-0.14</td>
<td>-0.08</td>
</tr>
<tr>
<td>5y</td>
<td>0.36</td>
<td>0.17</td>
<td>-0.04</td>
<td>0.14</td>
<td>0.08</td>
<td>-0.10</td>
<td>0.09</td>
<td>-0.26</td>
<td>0.71</td>
<td>0.48</td>
</tr>
<tr>
<td>7y</td>
<td>0.34</td>
<td>0.27</td>
<td>0.15</td>
<td>0.01</td>
<td>0.00</td>
<td>-0.12</td>
<td>0.13</td>
<td>-0.54</td>
<td>0.00</td>
<td>-0.68</td>
</tr>
<tr>
<td>10y</td>
<td>0.31</td>
<td>0.30</td>
<td>0.28</td>
<td>-0.10</td>
<td>-0.06</td>
<td>0.01</td>
<td>0.03</td>
<td>-0.23</td>
<td>-0.63</td>
<td>0.52</td>
</tr>
<tr>
<td>30y</td>
<td>0.25</td>
<td>0.33</td>
<td>0.46</td>
<td>-0.34</td>
<td>-0.18</td>
<td>0.33</td>
<td>-0.09</td>
<td>0.52</td>
<td>0.26</td>
<td>-0.13</td>
</tr>
</tbody>
</table>

**Table 4.9** Factor loadings for US Treasury data.
Table 4.10  Standard deviation of factor scores (basis points).

<table>
<thead>
<tr>
<th>PCI</th>
<th>PC2</th>
<th>PC3</th>
<th>PC4</th>
<th>PC5</th>
<th>PC6</th>
<th>PC7</th>
<th>PC8</th>
<th>PC9</th>
<th>PC10</th>
</tr>
</thead>
<tbody>
<tr>
<td>17.49</td>
<td>6.05</td>
<td>3.10</td>
<td>2.17</td>
<td>1.97</td>
<td>1.69</td>
<td>1.27</td>
<td>1.24</td>
<td>0.80</td>
<td>0.79</td>
</tr>
</tbody>
</table>
Results (Tables 4.9 and 4.10 on page 101)

- The first factor is a roughly parallel shift (83.1% of variation explained)
- The second factor is a twist (10% of variation explained)
- The third factor is a bowing (2.8% of variation explained)
Results (continued) Figure 4.7 page 103

![Factor Loading vs Maturity (yrs)](image)

- Factor Loading vs Maturity (yrs)
  - PC1
  - PC2
  - PC3
Using PCA to calculate deltas

Table 7.9  Change in portfolio value for a 1-basis-point rate move ($ millions).

<table>
<thead>
<tr>
<th>Rate:</th>
<th>1-year</th>
<th>2-year</th>
<th>3-year</th>
<th>4-year</th>
<th>5-year</th>
</tr>
</thead>
<tbody>
<tr>
<td>Change:</td>
<td>+10</td>
<td>+4</td>
<td>−8</td>
<td>−7</td>
<td>+2</td>
</tr>
</tbody>
</table>
Using PCA to calculate deltas

Using the data in Table 7.7, our delta exposure to the first factor (measured in millions of dollars per factor score basis point) is

\[ 10 \times 0.32 + 4 \times 0.35 - 8 \times 0.36 - 7 \times 0.36 + 2 \times 0.36 = -0.08 \]

and our delta exposure to the second factor is

\[ 10 \times (-0.32) + 4 \times (-0.10) - 8 \times 0.02 - 7 \times 0.14 + 2 \times 0.17 = -4.40 \]