Midterm Solutions
Macroeconomics I

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**Question 1.**

a. The state and control variables are as follows:

State variables: The quality of dental services in the current period \(s\) and previous period \((s-1)\).

Control Variable: The quality of dental services in the next period \((s+1)\).

The Bellman Equation will be:

\[
v(s, s_{-1}) = \arg \max_{s+1} \{(s - s_{-1}) \left(\frac{\bar{s}}{s}\right)^\rho - c(s, s_{+1}) + \beta v(s_{+1}, s)\} \quad (1)
\]

For the purpose of simplicity, we use this notation: \(u(s, s_{-1}) = (s - s_{-1}) \left(\frac{\bar{s}}{s}\right)^\rho\)

The first order condition with respect to \(s_{+1}\) is:

\[-c_2(s, s_{+1}) + \beta v_1(s_{+1}, s) = 0 \quad (2)\]

The Envelope Theorem for \(s_{-1}\) and \(s\) yields the following equations:

\[
s_{-1} : v_2(s, s_{-1}) = u_2(s, s_{-1}) \quad (3)
\]

\[
s : v_1(s, s_{-1}) = u_1(s, s_{-1}) - c_1(s, s_{+1}) + \beta v_2(s_{+1}, s) \quad (4)
\]

After doing some routine algebra, the final equation for solving for the steady state is as follows:

\[
\frac{1}{\beta} c_2(s_{-1}, s) = u_1(s, s_{-1}) - c_1(s, s_{+1}) + \beta u_2(s_{+1}, s) \quad (5)
\]

In the steady state, we have: \(s = s_{-1} = s_{+1} = \hat{s}\)

The partial derivatives in the steady state are:

\[
c_2(s_{-1}, s) = \frac{\mu}{\bar{s}} \quad (6)
\]

\[
c_1(s, s_{+1}) = -\frac{\mu}{\bar{s}} \quad (7)
\]

\[
u_1(s, s_{-1}) = \left(\frac{\bar{s}}{s}\right)^\rho \quad (8)
\]

\[
u_2(s_{+1}, s) = -\left(\frac{\bar{s}}{s}\right)^\rho \quad (9)
\]

Again with a little algebra, we drive the steady state quality of dental services:

\[
\hat{s} = \left(\frac{\beta}{\mu s^\rho}\right)^{\rho - 1} \quad (10)
\]
b. The social planner solves the following optimization problem:

$$\max_{s, \rho}\left\{ \frac{s^{1-\sigma}}{1-\sigma} - c(s, \hat{s}) \right\} \quad (11)$$

Social planner can solve this problem every period and come up with the proper policy parameters.

**Question 2.**

a. We can write the value function as follows:

$$v(w, \epsilon) = \max\left\{ v^{eat}(w, \epsilon), v^{noteat}(w, \epsilon) \right\} \quad (12)$$

The state variables are $w$ and $\epsilon$. The control variables is to eat or not.

b. Assume that there is a cutoff rule at $\bar{\epsilon}$, which means that the cake would be eaten if the realized $\epsilon$ is greater than this cutoff point and reserved for the next period otherwise. So, by this logic, we can rewrite the value function:

$$v(w, \epsilon) = \begin{cases} 
\beta Ev(\rho w, \epsilon') & \text{if } \epsilon < \bar{\epsilon} \\
\epsilon w^{1-\gamma} \quad & \text{if } \epsilon > \bar{\epsilon} 
\end{cases} \quad (13)$$

where, $Ev(\rho w, \epsilon')$ is:

$$Ev(\rho w, \epsilon') = \int_{\epsilon}^{+\infty} \frac{\epsilon' (\rho w)^{1-\gamma}}{1-\gamma} f(\epsilon') d\epsilon' \quad + \quad \beta \int_{-\infty}^{\bar{\epsilon}} Ev(\rho^2 w, \epsilon'') f(\epsilon') d\epsilon'$$

$$= \frac{(\rho w)^{1-\gamma}}{1-\gamma} \int_{\epsilon}^{+\infty} \epsilon' f(\epsilon') d\epsilon' \quad + \quad \beta Ev(\rho^2 w, \epsilon'').prob(\epsilon' < \bar{\epsilon}) \quad (14)$$

Note that $v(w, \epsilon)$ is a linearly increasing function of $\epsilon$ when $\epsilon > \bar{\epsilon}$ and a constant when $\epsilon < \bar{\epsilon}$. Therefore, our guess here is that the plot of $v(w, \epsilon)$ with respect to $\epsilon$ is something like figure 1.

Let us assume that this function is continuous at $\epsilon = \bar{\epsilon}$. We also guess that the functional form of $Ev(\rho w, \epsilon')$ is:

$$Ev(\rho w, \bar{\epsilon}) = \bar{K} \frac{w^{1-\gamma}}{1-\gamma} \quad (15)$$

For verifying our guess, all we have to do is insert (15) into (14):

$$\bar{K} \frac{w^{1-\gamma}}{1-\gamma} = \frac{(\rho w)^{1-\gamma}}{1-\gamma} \int_{\epsilon}^{+\infty} \epsilon' f(\epsilon') d\epsilon' \quad + \quad \text{prob}(\epsilon' < \bar{\epsilon}) \bar{K} \frac{(\rho w)^{1-\gamma}}{1-\gamma} \quad (16)$$
With a little algebra, we obtain an expression for $\tilde{K}$:

$$\tilde{K} = \frac{\rho^{1-\gamma} \int_{\bar{\epsilon}}^{+\infty} \epsilon f(\epsilon')d\epsilon'}{1 - \beta \rho^{1-\gamma} \text{prob}(\epsilon' < \bar{\epsilon})}$$

(17)

It is obvious that $\tilde{K}$ is “not” a function of $\epsilon$ and so $Ev(\rho w, \epsilon')$ is in fact constant. From the continuity of the function $v(w, \epsilon)$ at $\epsilon = \bar{\epsilon}$, we can derive the cutoff amount of $\epsilon$:

$$\bar{\epsilon} = \frac{w^{1-\gamma}}{1 - \gamma} \frac{1}{1 - \beta \rho^{1-\gamma} \text{prob}(\epsilon' < \bar{\epsilon})} \Rightarrow \bar{\epsilon} = \frac{\beta \rho^{1-\gamma} \int_{\bar{\epsilon}}^{+\infty} \epsilon' f(\epsilon')d\epsilon'}{1 - \beta \rho^{1-\gamma} \text{prob}(\epsilon' < \bar{\epsilon})}$$

(18)

It is clear from (18) that the cutoff rule does not depend on $w$.

c. Some simple albeit tiresome(!) algebra yield the following result about the relationship between $\bar{\epsilon}$ and $\rho$:

$$\frac{\partial \bar{\epsilon}}{\partial \rho} > 0$$

(19)

Intuitively, this result makes sense. A decrease in $\rho$ would reduce $\bar{\epsilon}$ since the value of not eating the cake drops; cake shrinks slower and so it makes sense to eat the cake at a lower taste shock.