Final Exam Solutions
Macroeconomics I

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Question 1.

a. The state and control variables are as follows:
  State variables: Number of hens consumed ($h$), eggs ($e$) and number of alive hens ($k$).
  Control Variable: Number of alive hens in the next period ($k'$).

The Bellman Equation will be:

$$v(k) = \arg \max_{e,h,k'} \{u(e,h) + \beta v(k')\}$$ (1)

Subject to :

$$k' \leq (k - h)(1 - \delta) + f(k - e)$$ (2)

The functional form of $u(e,h)$ is defined in the problem. The first order condition with respect to $e,h$ and $k'$ is:

$$e : e^{-\psi} - \lambda f'(k - e) = 0$$ (3)

$$h : h^{-\psi} - \lambda(1 - \delta) = 0$$ (4)

$$k' : \beta v' - \lambda = 0$$ (5)

The Envelope Theorem for $k$ yields the following equations:

$$v' = \lambda f'(k - e) + \lambda(1 - \delta)$$ (6)

b. From (5) and (6) and using the fact that $f(\cdot) = x^\alpha$, we arrive at the following in the steady state:

$$\frac{1}{\beta} - (1 - \delta) = f'(k - e) = \alpha(k - e)^{\alpha - 1}$$ (7)

Combining (4) and (5) results in:

$$\left(\frac{h}{e}\right)^{\psi} = \frac{f'(k - e)}{1 - \delta} = \frac{\alpha}{1 - \delta} \frac{(k - e)^{\alpha}}{k - e}$$ (8)

With a little algebra, we have the following results:

$$k = \left[\left(\frac{1}{\alpha \beta} - \frac{1 - \delta}{\alpha}\right) \frac{1}{\alpha - 1} \equiv A\right] + e \Rightarrow k = A + e$$ (9)
\[
\frac{h}{e} = \left( \frac{1}{\beta(1-\delta)} - 1 \right)^\psi \equiv B
\]  
(10)

Using the resource constraint in the steady state and two above equations, we arrive at the steady state closed form of \(e, h\) and \(k\):

\[
e = \frac{A^\alpha - A}{1 - B(1 - \delta)}
\]
(11)

\[
h = \frac{B(A^\alpha - A)}{1 - B(1 - \delta)}
\]
(12)

\[
k = A + e
\]
(13)

c. For calibration of this problem, all we have to do is some simple algebra: elasticity of new chicken to number of eggs is 0.9 which is \(\alpha\).

\[
\text{number of new chickens} \quad \frac{\text{number of hens}}{k} = \frac{(k - e)^\alpha}{k} = 0.4.
\]

\[
h = 0.2 \quad \text{and} \quad \frac{e}{k} = 0.25
\]

Using the steady state equations, we have the following numbers for calibration:

\[
\psi = 2, \quad \beta = 0.81 \quad \text{and} \quad \delta = 0.25
\]

\[\text{Question 2.}\]

a. The Arrow-Debrue economy is defined as follows:

\[
\max_i \quad u_i(c) = \sum_{t=1}^{2} \sum_{s^t \in S^t} \pi_t(s^t)u(c^i_t(s^t)) \quad i = 1, 2
\]
(14)

Subject to:

\[
\sum_{t \epsilon S^t} p(s^t)e^i(s^t) = \sum_{t \epsilon S^t} p(s^t)c^i(s^t), \quad \forall s^t \epsilon S^t
\]
(15)

The market clearing condition is:

\[
\sum_{t \epsilon S^t} c^i(s^t) = \sum_{t \epsilon S^t} e^i(s^t), \quad \forall s^t \epsilon S^t
\]
(16)

b. The Pareto efficient allocation is:

\[
\max_2 \quad \sum_{t=1}^{2} \sum_{s^t \epsilon S^t} \pi_t(s^t)(\alpha u(c^1_t(s^t)) + (1 - \alpha)u(c^2_t(s^t))
\]
(17)
Subject to:
\[
\sum_{i=1}^{2} c^i(S^t) = \sum_{i=1}^{2} c^i(s^i), \quad \forall s^i \epsilon S^t
\]  
(18)

c.
\[\pi(s_1 = 1) = 0.5\]
\[\pi_2(s^2 = (2, 2)) = 0.5 \times 0.7 = 0.35\]
\[\pi_2(s^2 = (2, 1)) = 0.5 \times 0.3 = 0.15\]
0.7p_1 + 0.3p_2 = p_1
0.3p_1 + 0.7p_2 = p_2
⇒ 0.3p_1 = 0.3p_2 ⇒ p_1 = p_2
p_1 + p_2 = 1 ⇒ p_1 = p_2 = 0.5
d.
We can solve the social planner problem. The social planner maximizes the sum of the two agents' utility with a constraint on the endowments in each period. The utility function and endowment streams are given. All we have to do is solve the maximization problem in part (b) with the weights given to each agent be equal to 0.5. The first order condition with respect to each agent’s consumption would be:
\[
u'(c^i(s^i))\pi_1(s^i) = \lambda_1(s^i)
\]
(19)
From the symmetry of the problem, we can conjecture that \(\lambda_1 = \lambda_2\) and the consumption allocation would be as follows:
\[c^1(1) = c^2(1) = c^1(2) = c^2(2) = 1\]
\[c^1(1, 1) = c^2(1, 1) = c^1(1, 2) = c^2(1, 2) = ... = 1\]
It is straightforward to verify that these results are in fact correct and consistent with the resource constraint and endowment streams.
e.
If we solve the previous part with Arrow Securities, the price of these securities would be p(1)=1 and p(2)=1\(^1\). The call option is:
If \(s^1 = 1 ⇒ -p(1) + 0.7*p(1,1)*2 + 0.3*p(1,2)*2=0.16\)
If \(s^2 = 2 ⇒ -p(2) + 0.7*p(2,2)*2 + 0.3*p(2,1)*2=0.16\)
The call option price would be: 0.5*0.16+0.5*0.16=0.16
\(^{1}\)Try to solve it for a good exercise.