Game Theory
44812 (1393-94 2\textsuperscript{nd} term)

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Graduate School of Management and Economics
Sharif University of Technology

Spring 2015
Lectures:
- Saturday  Monday 15 - 16:30; Classroom 6.

Office Hours:
- Room 118; Monday 14 - 15; or by appointment (email: ffatemi@sharif.edu ).
Classes:

- One hour of weekly classes. Teacher: Mr. Saeed Shadkar Mr. Abdollah Farhoodi.

Evaluation:

- The evaluation of the course is based on problem sets (15 %), mid-term exam (40 %), and final exam (45 %). The weight for mid-term and final exam might change slightly considering the actual coverage of each of them. The mid-term and final exams are scheduled for 23/01/90 and 05/04/91 respectively.
Course Description:
Game Theory is a name for a collection of analytic tools which economists use to understand strategic interactions. The aim of this course is to learn how to analyze strategic behavior of rational decision makers. We say that decision making is strategic if it involves taking into account what other agents want, know, believe and do.

The focus of the course is on three equally important fronts:

- Students should get a good understanding (and some experience) of how to model a strategic environment as a game.
- Students will learn how to solve a game-theoretic model appropriately.
- The course will contain an overview of some classic applications of game theory mostly in economics.
Course Webpage:
URL: gsme.sharif.edu/ffatemi/Game_Theory.htm

Textbook:
- Osborne, Martin Ariel Rubinstein (OR); A Course in Game Theory; MIT Press.
- Osborne, Martin (OS); An introduction to Game Theory; Oxford University Press.

Other References:
- Gibbons, Robert; A Primer in Game Theory; Prentice Hall.
- Fudenberg Drew Jean Tirole; Game Theory; MIT Press.
Course Outline:

We intend to cover the following topics in this course:

- **Introduction (Os: 1  OR: 1)**
- **Strategic games with perfect information**
  - Definition and some examples (Os: 2.1-2.5  OR: 2.1)
  - Nash equilibrium (Os: 2.6-2.8; 2.10; 3  OR: 2.2-2.4)
  - Mixed strategy equilibrium (Os: 4  OR: 3.1-3.2)
  - Strictly competitive games and max-minimization (Os: 11  OR: 2.5)
  - Rationalizability, dominance, and iterated elimination of dominated actions (Os: 2.9; 12  OR: 4)
- **Strategic games with imperfect information**
  - Bayesian Nash equilibrium (Os: 9.1-9.3  OR: 2.6)
  - Applications: public good provision, auctions, juries, strategic voting, a model of knowledge (Os: 9.4-9.8  OR: 5)
Extensive games with perfect information
  - Definition, some examples, Nash equilibrium (Os: 5.1-5.3 OR: 6.1)
  - Subgame perfect equilibrium and backward induction (Os: 5.4-5.5 OR: 6.2)
  - Applications: ultimatum game, the hold-up game, agenda control, Stackelberg duopoly, buying votes (Os: 6-7 OR: 6.3-6.6)
  - Repeated games (Os: 14-15 OR: 8)
  - Bargaining games (Os: 16 OR: 7)

Extensive games with imperfect information (Os: 10 OR: 11.1)

Coalitional games and the core (Os: 8 OR: 13)

Introduction to mechanism design
**Game:** A situation in which intelligent decisions are necessarily interdependent. A situation where utility (payoff) of an individual (player) depends upon her own action, but also upon the actions of other agents.

How does a rational individual choose?

My optimal action depends upon what my opponent does but his optimal action may in turn depend upon what I do.
Examples:
- Matching Pennies
- Rock, Scissors, Paper
- Dots Crosses (Tick-Tack-Toe)
- Meeting in New York
- Prisoners Dilemma
Game Components:

Players: Rational agents who participate in a game and try to maximise their payoff.

Strategy (action): An action which a player can choose from a set of possible actions in every conceivable situation.

Strategy profile: A list of strategies including one strategy for each player.

Order of play: Shows who should play when? At each history, which player should to choose his action. Player may move simultaneously or sequentially. Game might have one round which players act simultaneously.
**Information set:** What players know about previous actions when it is their turn?

**Outcome:** For each set of actions (at each terminal history) by the players what is outcome of the game.

**Payoff:** The utility (payoff) that a player receives depending on the strategy profile chosen (outcome). Players need not be concerned only with money and could be altruistic, or could be concerned not to violate a norm.
Each player seeks to maximize his expected payoff (in short, is rational). Furthermore he knows that every other player is also rational, and knows that every other player knows that every player is rational and so on (rationality is common knowledge).

The theory of rational choice:

*The action chosen by a decision maker is as least as good as any other available action (according to his preferences).*
Definition (Os 13.1)

A **strategic game** consists of

- a set of players
- for each player, a set of actions
- for each player, ordinal preferences (a payoff function) over the set of action profiles.

Note: The preferences can be ordinal.

Time is absent form this definition; players play simultaneously. An action can be a contingent plan that is why it is sometimes called a strategy. But since in this setting the time is absent then the two are equivalent.
Strategic form of a simultaneous move game:

<table>
<thead>
<tr>
<th></th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$u_r^{AC}$, $u_c^{AC}$</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
**Example:** Prisoners Dilemma

<table>
<thead>
<tr>
<th>Prisoner 1</th>
<th>Confess</th>
<th>Don't Confess</th>
</tr>
</thead>
<tbody>
<tr>
<td>Confess</td>
<td>-4, -4</td>
<td>-1, -10</td>
</tr>
<tr>
<td>Don't Confess</td>
<td>-10, -1</td>
<td>-2, -2</td>
</tr>
</tbody>
</table>

Since the preferences are ordinal it is equivalent to:

<table>
<thead>
<tr>
<th>Prisoner 1</th>
<th>Confess</th>
<th>Don't Confess</th>
</tr>
</thead>
<tbody>
<tr>
<td>Confess</td>
<td>1, 1</td>
<td>3, 0</td>
</tr>
<tr>
<td>Don't Confess</td>
<td>0, 3</td>
<td>2, 2</td>
</tr>
</tbody>
</table>
**Example:** Working on a Project

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Workhard</strong></td>
<td><strong>Don'tBother</strong></td>
</tr>
<tr>
<td>2, 2</td>
<td>0, 3</td>
</tr>
<tr>
<td>3, 0</td>
<td>1, 1</td>
</tr>
</tbody>
</table>
**Example:** The money sharing game

<table>
<thead>
<tr>
<th></th>
<th>Share</th>
<th>Grab</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>A</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Share</td>
<td>$\frac{M}{2}$</td>
<td>$\frac{M}{2}$</td>
</tr>
<tr>
<td>Grab</td>
<td>$M$</td>
<td>$2$</td>
</tr>
<tr>
<td><strong>B</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Share</td>
<td>$0$</td>
<td>$M$</td>
</tr>
<tr>
<td>Grab</td>
<td>$0$</td>
<td>$0$</td>
</tr>
</tbody>
</table>
Example: Battle of Sexes
Two players are to choose simultaneously whether to go to the cinema or theatre. They have different preferences, but they both would prefer to be together rather than go on their own.

<table>
<thead>
<tr>
<th></th>
<th>Cinema</th>
<th>Theatre</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Husband</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cinema</td>
<td>2,1</td>
<td>0,0</td>
</tr>
<tr>
<td>Theatre</td>
<td>0,0</td>
<td>1,2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Cinema</th>
<th>Theatre</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Wife</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cinema</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Theatre</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
**Example:** Duopoly

Two firms competing in a market should decide simultaneously whether to price high or low.

<table>
<thead>
<tr>
<th></th>
<th>Firm 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>High</td>
<td></td>
</tr>
<tr>
<td>Low</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>High</th>
<th>Low</th>
</tr>
</thead>
<tbody>
<tr>
<td>Firm 1</td>
<td>1000, 1000</td>
<td>-200, 1200</td>
</tr>
<tr>
<td></td>
<td>1200, -200</td>
<td>600, 600</td>
</tr>
</tbody>
</table>

This game is equivalent to Prisoners Dilemma.
### Strategic form:

<table>
<thead>
<tr>
<th></th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Grand Central</td>
</tr>
<tr>
<td>Player 1</td>
<td>100, 100</td>
</tr>
<tr>
<td></td>
<td>0, 0</td>
</tr>
</tbody>
</table>
Representation of a Game

Extensive form:

\[ \begin{array}{c}
\text{Player 1} \\
\text{Grand Central} \quad \text{Empire State} \\
\text{Empire State} \quad \text{Grand Central} \\
(100, 100) \quad (0, 0) \quad (0, 0) \quad (100, 100)
\end{array} \]
Definition

A **Nash Equilibrium** is an action profile \( a^* \in A \) with the property that no player \( i \) can do better by choosing an action different from \( a^*_i \), given that all other players stick to \( a^*_{-i} \).

(we show an action profile for player \( i \)'s opponents by \( a_{-i} \in A_{-i} \))

Definition (Os 23.1)

The action profile \( a^* \in A \) in a strategic game is a **Nash Equilibrium** if for every player \( i \):

\[
 u_i(a^x) \geq u_i(a^*_i, a^*_{-i}) \text{ for every action } a_i \text{ of player } i
\]

Where \( u_i(.) \) is a payoff function representing player \( i \)'s preferences.
Example: Prisoners Dilemma

<table>
<thead>
<tr>
<th></th>
<th>Confess</th>
<th>Don’t Confess</th>
</tr>
</thead>
<tbody>
<tr>
<td>Confess</td>
<td>−4, −4</td>
<td>−1, −10</td>
</tr>
<tr>
<td>Don’t Confess</td>
<td>−10, −1</td>
<td>−2, −2</td>
</tr>
</tbody>
</table>

(NC, NC) is not a Nash eq. Prisoner 1 would prefer to play C instead of NC.

(NC, C) is not a Nash eq. Prisoner 1 would prefer to play C instead of NC. (a similar argument for (C, NC)).

(C, C) is the only Nash eq. of the game. No one has incentive to deviate from this strategy profile.
Example: Battle of Sexes

\[
\begin{array}{c|cc}
& \text{Cinema} & \text{Theatre} \\
\hline
\text{Cinema} & 2, 1 & 0, 0 \\
\text{Theatre} & 0, 0 & 1, 2 \\
\end{array}
\]

(Cinema, Cinema) and (Theatre, Theatre) are the two NEs of this game.
Example: Matching Pennies

\[
\begin{array}{c|cc}
& \text{Head} & \text{Tail} \\
\text{Head} & -1, 1 & 1, -1 \\
\text{Tail} & 1, -1 & -1, 1 \\
\end{array}
\]

This game has no NE.

We will return to this game to study the likely outcome of the game.
Example: Stag Hunt

Two hunters should help each other to be able to catch a stag. Alternatively each can hunt a hare on their own (the initial idea of this game is from Jean-Jacques Rousseau). The payoffs are:

<table>
<thead>
<tr>
<th></th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Stag</strong></td>
<td>0, 1</td>
</tr>
<tr>
<td><strong>Hare</strong></td>
<td>1, 1</td>
</tr>
</tbody>
</table>

(Stag, Stag) and (Hare, Hare) are the two NEs of this game.
Stag Hunt with $n$ Hunters (Os Exercise 30.1)

There are $n$ hunters. Only $m$ hunters are enough to catch a stag where $2 \leq m < n$. Assume there is only a single stag.

What is the NE of the game if:

a) Each hunter prefers the fraction $\frac{1}{n}$ of the stag to a hare.

b) Each hunter prefers the fraction $\frac{1}{k}$ of the stag to a hare ($m \leq k \leq n$), but prefers a hare to any smaller fraction of the stag.
Example: a different version of battle of sexes (a coordination game)

<table>
<thead>
<tr>
<th></th>
<th>Cinema</th>
<th>Theatre</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cinema</td>
<td>2, 2</td>
<td>0, 0</td>
</tr>
<tr>
<td>Theatre</td>
<td>0, 0</td>
<td>1, 1</td>
</tr>
</tbody>
</table>

(Cinema, Cinema) and (Theatre, Theatre) are the two NEs of this game.

Which eq. is the most likely outcome of this game? Why?
**Example:** Chicken Game (Hawk-Dove game)

<table>
<thead>
<tr>
<th></th>
<th>Driver 2</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Swerve</strong></td>
<td></td>
</tr>
<tr>
<td>Driver 1</td>
<td>0, 0</td>
</tr>
<tr>
<td></td>
<td>2, −1</td>
</tr>
<tr>
<td><strong>Straight</strong></td>
<td></td>
</tr>
</tbody>
</table>

NE: (Sw, St) and (St, Sw).

Game is symmetric but the pure NE is asymmetric.
Why NE?

- Why NE makes sense as an equilibrium concept for a given game?
- Why should we believe that players will play a Nash equilibrium?

a) NE as a consequence of rational inference.

b) Self enforcing agreement: If the players reach an agreement about how to play the game, then a necessary condition for the agreement to hold up is that the strategies constitute a Nash equilibrium. Otherwise, at least one player will have an incentive to deviate.
c) Any prediction about the outcome of a non-cooperative game is self-defeating if it specifies an outcome that is not a Nash equilibrium. (NE as a necessary condition if there is a unique predicted outcome to a game).

d) Result of process of adaptive learning: Suppose players play a game repeatedly. The idea is that a reasonable learning process, if it converges, should converge to a Nash equilibrium (NE as a stable social convention).

e) If a game has a focal point, then it is necessary a NE. (Schelling (The strategy of conflict; 1960) introduced the concept of focal points).
Provision of a Public Good (Os Exercise 33.1)

Players: \( N = \{1, \ldots, n\} \)

Strategies: \( A_i = \{C, NC\} \)

- C: Contribute for the public good
- NC: Don’t contribute for the public good

Outcome: public good is provided if at least \( k \) people contribute.

\[
2 \leq k \leq n
\]

\[
\text{Outcome} = \begin{cases} 
1 & \text{if at least } k \text{ people contribute} \\
0 & \text{if less than } k \text{ people contribute}
\end{cases}
\]

Payoff:

\[
u_i(a_i = C; Out = 0) < u_i(a_i = NC; Out = 0) < u_i(a_i = C; Out = 1) < u_i(a_i = NC; Out = 1)
\]
What are the NE of this game?

- Is there a NE where more than $k$ players contribute?
- Is there a NE where exactly $k$ players contribute?
- Is there a NE where less than $k$ players contribute?
Best response function of player denotes a players best reaction (utility maximising reaction) to any strategy profile chosen by other players.

**Definition**

Player is **best response function (correspondence)** in a strategic game is the function that assigns to each $a_{-i} \in A_{-i}$ the set:

$$BR_i(a_{-i}) = \{ a_i \in A_i : u_i(a_i, a_{-i}) \geq u_i(a'_i, a_{-i}); \forall a'_i \in A_i \}$$
Proposition (Os 36.1)

The action profile $a^*$ is a Nash equilibrium of a strategic game if and only if every player's action in this profile is a best response to the other player's actions ($a^*_i$):

$$a^*_i \in BR_i(a^*_{-i}) \quad \text{for} \quad i = 1, \ldots, N$$

All individual strategies in $a^*$ are best responses to each other. Recall: A **Nash Equilibrium** is an action profile $a^* \in A$ with the property that no player $i$ can do better by choosing an action different from $a^*_i$, given that all other players stick to $a^*_{-i}$.
Using BR function to find NE

a) Find the BR function of each player

b) Find the action profiles that satisfy:

\[ a_i^* \in BR_i(a_{-i}^*) \quad \text{for} \quad i = 1, \ldots, N \]
Example: Matching Pennies

\[
\begin{array}{cc|cc}
\text{Head} & \text{Tail} \\
\hline
\text{Head} & -1, 1 & 1, -1 \\
\text{Tail} & 1, -1 & -1, 1 \\
\end{array}
\]
Example: Chicken Game

<table>
<thead>
<tr>
<th></th>
<th>Driver 1</th>
<th>Driver 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Swerve</td>
<td>Straight</td>
</tr>
<tr>
<td>0, 0</td>
<td>-1, 2</td>
<td></td>
</tr>
<tr>
<td>2, -1</td>
<td>-5, -5</td>
<td></td>
</tr>
</tbody>
</table>
Cournot Duopoly

Two firms, 1 and 2, producing a homogeneous good
The inverse demand function for the good is \( P = 10 - \frac{1}{10} Q \)
They choose quantity \( q_i \geq 0 \) simultaneously
For simplicity suppose marginal costs are zero.
Total quantity \( Q = q_1 + q_2 \) is placed on the market and determines the price.
Firm 1’s Profit is:

\[ \pi_1 = q_1 \cdot P = q_1 (10 - 0.1q_1 - 0.1q_2) = 10q_1 - 0.1q_1^2 - 0.1q_1q_2 \]

Suppose firm 2 fixes his production level at \( \hat{q}_2 \); then the best response by firm 1 should satisfy the first order condition:

\[ \frac{\pi_1}{\partial q_1} = 0 \Rightarrow 10 - 0.2q_1 - 0.1\hat{q}_2 = 0 \quad or \quad q_1 = 50 - 0.5\hat{q}_2 \]

Then the BR function for firm 1 is:

\[ q_1 = BR_1(\hat{q}_2) = 50 - 0.5\hat{q}_2 \]
And similarly for firm 2:
\[ q_2 = 50 - 0.5q_1 \]

To find the NE we have to set \( q_2 = \hat{q}_2(q_1) \)
\[ q_2 = 50 - 0.5(50 - 0.5q_2) \]

And the only NE is:
\[ q_1^C = q_2^C = \frac{100}{3} \]
Remember the monopoly quantity is: \( q^M = 50 \).

Easy to calculate that:

\[
P^M = 5, \quad \pi^M = 250
\]

and

\[
P^C = \frac{100}{3}, \quad \pi_1^C = \pi_2^C = 111.1
\]
Example: Bertrand Duopoly

Same context, but firms choose prices. Prices can be continuously varied, i.e. $p_i$ is any real number. Firms have the same marginal cost of $mc$.

If prices are unequal, all consumers go to lower price firm.

If equal, market is shared.

1) There is a Nash equilibrium where $p_1 = p_2 = mc$: None of the firms has incentive to deviate from this strategy.

2) There is no other Nash equilibrium in pure strategies.