Extensive Games with Imperfect Information

Definition (Os 314.1): An extensive game with imperfect information consists of

- a set of players \( N \)
- a set of terminal histories \( H \); no sequence is a proper subhistory of any other sequence
- a player function \( P(h) \) that corresponds either a player or the nature for every sequence that is a proper subhistory of some terminal history
- A probability distribution for every history which has assigned to the nature over the actions available at that history
- For each player an information partition
- for each player, preferences over the set of lotteries over \( H \)
Note: A simultaneous move game can be modelled as an extensive form.

Like this example of BoS
Definition (Os 318.1): A pure strategy of player $i$ in an extensive game is a function that assigns to each of information sets $I_i$ an action in the set of actions available at that information set $A(I_i)$.

Before going further let’s have a look at some examples of extensive games with imperfect information
• **Signaling Games**

Bayesian game with dynamic structure.

One informed party, one uninformed party.

Informed party tries to signal her type to uninformed.

Set of types of the sender $\Theta = \{\theta_1, \theta_2, \ldots, \theta_K\}$;

Prior probability $P = (p_1, p_2, \ldots, p_K)$;

$p_k$: the probability that sender is of type $\theta_k$ where:
\[ 0 \leq p_k \leq 1 \quad \text{and} \quad \sum_k p_k = 1 \]

Set of actions (messages) that the sender has is \( M = (m_1, m_2, \ldots, m_H) \)

Set of actions that the receiver has is \( A = (a_1, a_2, \ldots, a_L) \)

Utility functions:
Sender: $U(\theta_k, m_h, a_l)$ or for simplicity $U(\theta, m, a)$
Receiver: $V(\theta, m, a)$

Game:

Nature (Chance) chooses sender's type $\theta$ according to the probability distribution $P$

Sender observes his own type, but receiver does not.

Sender chooses a message $m$

Receiver observes the chosen message and chooses an action $a$
Payoffs are $U(\theta, m, a)$ and $V(\theta, m, a)$

Job market example (Spence Model):

Two type of workers:

\[ 0 < \theta_L < \theta_H \]
\[ 0 < \lambda = \text{Prob}(\theta = \theta_H) < 1 \]

Each worker can obtain some education prior to entering the job market, which has these properties:
Obtaining the education is costly

Education has no effect on worker’s ability (productivity)

Education level is observable

The cost of education level \( e \geq 0 \) for type \( \theta \):

\[
c(e, \theta)
\]

Where for \( \forall e, \theta \):

\[
c(0, \theta) = 0
\]

\[
\frac{\partial c(e, \theta)}{\partial e} > 0 \quad , \quad \frac{\partial^2 c(e, \theta)}{\partial e^2} > 0
\]

\[
\frac{\partial c(e, \theta)}{\partial \theta} < 0 \quad , \quad \frac{\partial^2 c(e, \theta)}{\partial e \partial \theta} < 0
\]
Worker’s utility if she obtains the education level of $e$ and get wage of $w$:

$$u(w, e|\theta) = w - c(e, \theta)$$

The sequence of the game:

1) Nature determines the type of worker

2) Observing her type; the worker decides about her level of education

3) Observing the worker’s education, but not her type; firm makes wage offer
4) The worker decides whether to accept the offer or remain unemployed

Equilibrium

A pure strategy for the sender specifies a message for each possible type she could be:

\[ \sigma: \Theta \rightarrow M \]
A pure strategy for the receiver specifies an action for each possible message she could receive:

$$\rho: M \rightarrow A$$

$(\sigma, \rho)$ is an "equilibrium" if:

$\sigma$ is optimal given $\rho$

Since all types $\theta$ have positive probability, this requires that for every $\theta$:

$$U\left(\theta, \sigma(\theta), \rho(\sigma(\theta))\right) \geq U(\theta, m, \rho(m)) \quad ; \quad \forall m \in M$$
How are we to define optimality for the receiver?

Payoff is $V(\theta, m, a)$

She knows $m$ and has to choose $a$; but does not know $\theta$

She forms beliefs (regarding the type of sender) on observing $m$

For any $m$; beliefs are (Vector of posterior beliefs)

$$
(\mu(\theta|m)) = (\mu(\theta_1|m), \mu(\theta_2|m), \ldots, \mu(\theta_K|m))
$$

$$
\mu(\theta_k|m) \geq 0 \quad \text{and} \quad \sum_k \mu(\theta_k|m) = 1
$$
Chooses \( a \) to maximize

\[
\sum_k \mu(\theta_k | m) \, V(\theta_k, m, a)
\]

A belief system \( \mu \) for the receiver is a collection of beliefs

\[
(\mu(\theta | m))_{m \in M}
\]

i.e. beliefs specified for every message in \( M \):

\( \rho \) is optimal given belief system \( \mu \) if \( \rho(m) \) maximizes:
$$\sum_{k} \mu(\theta_k|m) V(\theta_k, m, a) \quad \text{for} \quad \forall m \in M$$

How are beliefs formed?

Bayes rule using sender's strategy $\sigma$

Suppose $m$ is sent by some types in $\Theta$ under $\sigma$

Let
\[ D(\theta_k, m; \sigma) = \begin{cases} 1 & \text{if } \sigma(\theta_k) = m \\ 0 & \text{if } \sigma(\theta_k) \neq m \end{cases} \]

\[ \mu(\theta_\mathring{k} | m) = \frac{p(\theta_\mathring{k}).D(\theta_\mathring{k}, m; \sigma)}{\sum_k p(\theta_k).D(\theta_k, m; \sigma)} \]

This is defined if and only if the denominator is not zero

i.e. if \( m \) is sent by some type in \( \Theta \)

Beliefs depend upon the prior, \( p \); and upon the sender's strategy \( \sigma \)
If $m$ is not sent by any type in $\Theta$; then $\mu(\theta_k|m)$ can be arbitrary.

Define: Assessment = strategy profile + belief system

A **weak sequential equilibrium** is an assessment that satisfies sequential rationality and weak consistency.
A belief system is weakly consistent given $\sigma$ if beliefs are given by Bayes rule wherever possible.

Sequential rationality: given a player's beliefs and the strategies of other players, she must be maximizing his own payoff

- **Example: Signaling Game**

  Follow Osborne Chapter 10.5
• Example: Spence’s Job Market Model

Follow Osborne Chapter 10.7
Example: Cheap Talk

Model of advice

Sender (e.g. adviser) is informed about the state of the world
Receiver (e.g. minister) is uninformed

Sender and receiver have some conflict of interest

E.g. sender is more right wing than receiver

"State of the world" $t$ is uniformly distributed on $[0; 1]
Receiver's optimal action is $a = t$

Sender's optimal action is $a = t + b$

$b$ measures conflict of interest.

$$M = T$$

$$A = [0; 1 + b]$$

Receiver's payoff

$$V = -(a - t)^2$$

Sender's payoff

$$U = -[a - (t + b)]^2 ; b > 0$$
There is no equilibrium where the sender reports the true state of the world. Why?

There is no equilibrium where the sender reports any increasing or decreasing function of the true state of the world. Why?

In general an informative equilibrium cannot be fully informative.

Suppose each type of sender sends different message $m(t) \neq m(t')$ if $t$ different from $t'$

The function $m(.)$ is one to one and therefore invertible.

Receiver will choose $a(m) = t^{-1}(m)$

But then sender of type $t$ should send $m(t + b)$. 
Always exists a babbling equilibrium:

For every $t$; every message is sent with equal probability, and

$$a(m) = 0.5 \quad \forall m$$

Given sender's strategy, for every message, receiver's same as prior optimal to choose 0.5

Given that receiver does not condition on message, all messages yield same payoff to sender.

So optimal to send any message.
Partition equilibria:

e.g.
All senders in interval \([0; x]\) randomize uniformly on \(m \in [0 ; x]\)
All senders in interval \((x; 1]\) randomize uniformly on \(m \in (x ;1]\)

On receiving \(m \in [0 ; x]\) receiver believes that \(t\) is uniform on \([0; x]\)
Optimal to choose \(a = x/2\)

On receiving \(m \in (x ;1]\) receiver believes that \(t\) is uniform on \((x ;1]\)
Optimal to choose \(a = (x+1)/2\)
Incentive condition should be satisfied for both $t \in [0 ; x]$ and $t \in (x ; 1]$: 

Claim: if $x = 0.5 - 2b$, then incentive conditions are satisfied all $t$

No unsent messages, so beliefs fully determined by the Bayes rule.

Remember $b$ measures conflict of interest between sender and receiver:

If $b \geq 0.25$; no information transmission possible.
If $b$ is small, more information transmission possible.
Partition equilibria:

Interval $[0; 1]$ is partitioned into several subintervals.

Sender sends same messages in each subinterval, different messages across subintervals.

If $b < 1/12$; equilibria with 3 subintervals possible.
What is the expected utility of sender and receiver in these equilibria?

Ex-ante expected utility, before $t$ is realized.

With no information transmission (or babbling)

$$E(U) = -\int_{0}^{1} (\tau + b - 0.5)^2 \, d\tau$$

$$E(V) = -\int_{0}^{1} (\tau - 0.5)^2 \, d\tau$$
In equilibrium with meaningful communication, expected utility of both parties higher:

E.g. in two subinterval equilibrium,

\[ E(U) = -\int_{0}^{0.25} (\tau + b - 0.125)^2 d\tau - \int_{0.25}^{1} (\tau + b - 0.625)^2 d\tau \]

\[ E(V) = -\int_{0}^{0.25} (\tau - 0.125)^2 d\tau - \int_{0.25}^{1} (\tau - 0.625)^2 d\tau \]
Delegation

Suppose receiver delegates decision to sender

Decision will be biased
But possibly more aligned with information

Assume commitment: receiver cannot revise decision taken by sender

Sender will choose \( a = t + b \) for all \( t \)

Loss of sender = 0
Loss of receiver = \( b^2 \) for all \( t \)

If \( b \) is not too large, this is better for receiver than cheap talk
Delegation better than cheap talk if bias is small

If bias is large, no communication may be possible and also delegation maybe worse than just taking uninformed decision

Delegation, while restricting choice of sender might be better

e.g. do not permit actions greater than 1

Problem of commitment: receiver may want to overturn the decision once it has been made