Repeated Games

Basic lesson of prisoner’s dilemma:

In one-shot interaction, individual’s have incentive to behave opportunistically

Leads to socially inefficient outcomes

In reality; some cases of prisoner’s dilemma end in cooperation

What happens if interaction is repeated?
If you cooperate today, I will reward you tomorrow (by cooperating)

If you defect today, I will punish you tomorrow (by defecting)

This provides an incentive to cooperate today

Under what conditions does this argument work?

Focus on subgame perfect equilibrium
Finitely repeated games

Consider this prisoner’s dilemma situation:

<table>
<thead>
<tr>
<th></th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>1, 1</td>
<td>-L, 1 + G</td>
</tr>
<tr>
<td>D</td>
<td>1 + G, -L</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

Suppose that prisoner’s dilemma is repeated twice.

After play in period 1, players observe chosen action profile, a₁.

Simultaneously choose actions in period 2.
Payoffs in the overall game equal the discounted sum of payoffs in two stages

1 unit payoff in period 2 worth $\delta < 1$ when evaluated at period 1

Possible histories at end of period 1: \((C;C); (C;D); (D;C); (D;D)\)

A strategy for a player must specify what she does at every history also what she does in period 1.

For example, one strategy is \(C\) at \(t = 1\); \(C\) if \((C;C)\); \(D\) if not \((C;C)\)

Solve backwards, for a subgame perfect equilibrium

Let \(a^1\) be arbitrary: \((D;D)\) unique NE in this subgame
So payoff is period 2 equals 1, independent of period one actions.

Total payoff to 1 as a function of period 1 actions (µ the payoff from first round):

\[
\begin{array}{c|cc}
 & C & D \\
\hline
C & 1 + \delta \mu & -L + \delta \mu \\
D & 1 + G + \delta \mu & 0 + \delta \mu \\
\end{array}
\]

Since D strictly dominates C; unique Nash equilibrium in stage 1 augmented game is to (D;D)

Unique subgame perfect equilibrium strategy (for each player)

D at t = 1; D at t = 2 after every action profile a₁
Suppose the game is repeated for $T$ periods

Strategy has to specify action in every period after every history

1 unit of payoff in period $t$ is worth $\delta^{t-1}$ in period 1.
**Proposition:** Unique subgame perfect equilibrium: each player plays D in every period and after every history.

In period T; after any history h; D is strictly dominant.

So (D;D) is played after every history, in period T.

In T - 1; you cannot affect payoff in period T.

So after any history, D is strictly dominant, and (D;D) must be played.
Proof by induction:

i) \((D;D)\) is played after every history in period \(T\)

ii) Assume that \((D;D)\) is played after every history in periods \(t + 1; t + 2; \ldots; T\)

Then \((D;D)\) must be played in any SPNE in period \(t\)

By the induction hypothesis, what happens in period \(t\) does not affect payoffs in periods after \(t\)

So strictly dominant to play \(D\) at any history, and \((D;D)\) must be played.
Infinitely repeated games

Suppose the game is infinitely repeated

Expected payoffs well defined since $\delta < 1$

Strategy for player $i$ ($s_i$):

must specify action in period $t$ for any history $(a_1; a_2; \ldots; a_T)$

Subgame perfect equilibrium strategy profile $(s_1; s_2)$ must be a Nash equilibrium after any history
Some example of strategies:

Always D \{play D in every period after every history\}

Always C

Grim Trigger: Divide histories into two classes

Good histoires \((h)\): either \(t = 1\) or if \(t > 1\); both players have played C in every period.

Bad histories \((\tilde{h})\): \(t > 1\) and some player has played D in some period.

GT: play C at a good history, D at bad history.
Limited punishment:

At \( t = 1 \) play C

At \( t > 1 \); play C as long as your opponent played C in previous periods

After observing a D play D for a number of periods then return to playing

Tit-for-tat: Divide histories into two classes

The length of punishment depends on the behaviour of the opponent
(Always D, Always D) is a subgame perfect equilibrium (SPNE)

(Always C, Always C) is a not a SPNE

In general, verifying that a strategy profile is a SPNE is complicated

A player can deviate from the strategy in complicated ways

We need to make sure that the strategy cannot be improved upon by arbitrary deviations
One step deviation from a strategy $s_i$

suppose $s_i$ prescribes action $a$ in period $t$ at some history $h^{t-1}$

A one step deviation from $s_i$ chooses some other action $a'$ at $h^{t-1}$; but then plays as $s_i$ does in every subsequent history

One-step deviation property: no player can increase her payoff at some history by a one-step deviation, given the strategies of the other players.
**Proposition:** Suppose that we have finitely repeated game or an infinitely repeated game with $\delta < 1$: $(s_1; s_2)$ is a subgame perfect equilibrium if and only if it satisfies the one-step deviation property at every history.

Is $(GT, GT)$ a SPNE for the Game we seen above?

What about $(Tit-for-tat, Tit-for-tat)$?