• **Rationalizability**

In a NE, every player should play the best response according to his belief about other players’ behavior and this belief should be true.

Rationalizability offers an alternative and weaker condition than the NE where it is not necessary for players to have the right belief.

**Definition:** A *belief* of player $i$ about the other players’ actions is a probability distribution $\mu_i$ over the set $A_{-i}$. Then, $\alpha_i \in \Delta(A_i)$ is **rational for the belief** $\mu_i$ or a **best response to** $\mu_i$, if it maximizes the expected payoff of the player $i$ given his belief:

$$\sum_{a_{-i} \in A_{-i}} \mu_i(a_{-i}) U_i(\alpha_i, a_{-i})$$
Example:

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>C</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>0.7</td>
<td>2.5</td>
<td>7.0</td>
</tr>
<tr>
<td>M</td>
<td>5.2</td>
<td>3.3</td>
<td>5.2</td>
</tr>
<tr>
<td>D</td>
<td>7.0</td>
<td>2.5</td>
<td>0.7</td>
</tr>
</tbody>
</table>

The unique NE of the game: (M, C)

U is rational, if P1 believes that his opponent plays R.

R is rational, if P2 thinks that his opponent plays D.

D is rational, if P1 believes that his opponent plays L.

L is rational, if P2 thinks that his opponent plays U.
**Definition:** The action $a_i^* \in A_i$ is *rationalizable* if for each player $j$ there is a $Z_j \subset A_j$ such that:

1) $a_i^* \in Z_i$

2) For every player $j$, every action $a_j \in Z_j$ is a best response to a belief of player $j$ that assigns positive probability only to list s of actions in $Z_{-j}$. 
Immediate results of this definition are:

**Proposition:** Every action which appears with positive probability in a mixed strategy NE is rationalizable.

**Proposition:** A strictly dominated action is not rationalizable. More precisely, an action is a never-best response if and only if it is strictly dominated.

Then, an easy way to find rationalizable actions is to iteratively eliminate strictly dominated actions. Furthermore, it can be shown that the unique set of rationalizable actions is equivalent to the set of actions surviving iterative elimination of strictly dominated actions.
Strategic Games with Imperfect Information

Let’s start with an example (a different version of BoS):

\[
\begin{align*}
\pi &= \%50 \\
1 - \pi &= \%50
\end{align*}
\]

<table>
<thead>
<tr>
<th></th>
<th>Cinema</th>
<th>Theatre</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Husband</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cinema</td>
<td>2, 1</td>
<td>0, 0</td>
</tr>
<tr>
<td>Theatre</td>
<td>0, 0</td>
<td>1, 2</td>
</tr>
</tbody>
</table>

**Good day:**

*W* wishes to meet *H*

<table>
<thead>
<tr>
<th></th>
<th>Cinema</th>
<th>Theatre</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Wife</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cinema</td>
<td>2, 0</td>
<td>0, 2</td>
</tr>
<tr>
<td>Theatre</td>
<td>0, 1</td>
<td>1, 0</td>
</tr>
</tbody>
</table>

**Bad day:**

*W* likes to avoid *H*
Husband’s expected payoff conditional on Wife’s actions in a good or bad day:

<table>
<thead>
<tr>
<th></th>
<th>(C, C)</th>
<th>(C, T)</th>
<th>(T, C)</th>
<th>(T, T)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Husband</strong></td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td><strong>Wife</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>C</strong></td>
<td>0</td>
<td>0.5</td>
<td>0.5</td>
<td>1</td>
</tr>
<tr>
<td><strong>T</strong></td>
<td>0</td>
<td>0.5</td>
<td>0.5</td>
<td>1</td>
</tr>
</tbody>
</table>

In order to find a pure strategy Nash equilibrium we need to find a strategy profile consisting of three actions one for the Husband and two for each type of Wife.
H plays C  ⇒  W’s BR is (C , T)
H plays T  ⇒  W’s BS is (T , C)

W plays (C , C)  ⇒  H’s BR is C
W plays (C , T)  ⇒  H’s BR is C
W plays (T , C)  ⇒  H’s BR is C
W plays (T , T)  ⇒  H’s BR is T

Is there a NE (When one’s action is BR to its BR)?

[C , (C , T)]
Definition (Os 13.1): A strategic game with imperfect information (Bayesian Game) consists of

- a set of **players** \( N \)
- a set of **states** \( \Omega \)
- for each player, a set of **actions** \( A_i \) (define \( A = \times_{i \in N} A_i \))
- for each player, a set of **signals** that she may receive \( T_i \)
- for each player, a **signal function** that corresponds a signal for each state \( \tau_i: \Omega \rightarrow T_i \)
- for each player, a **belief** for each signal she receives about the states consistent with the signal \( p_i \left( \tau_i^{-1}(t_i) \right) > 0 \)
- for each player, a **Bernoulli payoff function** over the set of action profiles and states \( u_i(a, \omega) \) where \( a \in A \)

Note: Pure strategy for a player is a function \( s_i: T_i \rightarrow A_i \)
• Bayesian Nash Equilibrium

**Definition:** A *Bayesian Nash Equilibrium* is the Nash equilibrium of the following strategic game:

- **Players:** The set if all pairs \((i, t_i)\) where \(i\) is one of the players and \(t_i\) is one of the signals she may receive
- **Actions:** The set of actions of each player \((i, t_i)\) is the set of actions of player \(i\) in the Bayesian game
- **Preferences:** The expected payoff of player \((i, t_i)\): The expected payoff of player \(i\) when she receives a signal of \(t_i\) and chooses an action \(a_i\):

\[
u_i(a_i, t_i) = \sum_{\omega \in \Omega} \text{Prob}(\omega | t_i) \ u_i(a_i, \hat{a}_{-i}(\omega), \omega)
\]
**Example:** Find the BNE of the variant of BoS for any $\pi$:

Pure strategy equilibrium:

An action for player 1: $a_1$

&

A pair of actions for player 2: $a_2(t_1 = \omega_1); a_2(t_2 = \omega_2)$

Player 1’s action is optimal:

$$\pi u_1(a_1, a_2(\omega_1)) + (1 - \pi) u_1(a_1, a_2(\omega_2))$$

$$\geq \pi u_1(a'_1, a_2(\omega_1)) + (1 - \pi) u_1(a'_1, a_2(\omega_2)) ; \forall a'_1$$

(Player 1’s payoff does not depend upon state)
Player 2’s action is optimal at $\omega_1$:

$$u_2(a_1, a_2(\omega_1); \omega_1) \geq u_2(a_1, a_2'(\omega_1); \omega_1); \forall a_2'$$

Player 2’s action is optimal at $\omega_2$:

$$u_2(a_1, a_2(\omega_2); \omega_2) \geq u_2(a_1, a_2'(\omega_2); \omega_2); \forall a_2'$$

Player 2 knows the state, so the probability $\pi$ is not relevant for his calculation.
Let’s first check when \([C, (C, T)]\) is a NE:

For player 2 it is trivial that \((C, T)\) is the BR to \(C\).

For player 1, what is the BR to \((C, T)\):

\[
\begin{align*}
 u_1[C, (C, T)] &= \pi u_1(C, a_2(\omega_1) = C) + (1 - \pi) u_1(C, a_2(\omega_2) = T) \\
 u_1[C, (C, T)] &= \pi \times 2 + (1 - \pi) \times 0 = 2\pi
\end{align*}
\]

\[
\begin{align*}
 u_1[T, (C, T)] &= \pi u_1(T, a_2(\omega_1) = C) + (1 - \pi) u_1(T, a_2(\omega_2) = T) \\
 u_1[T, (C, T)] &= \pi \times 0 + (1 - \pi) \times 1 = 1 - \pi
\end{align*}
\]
Optimal to play B as long as $\pi \geq \frac{1}{3}$

If $\geq \frac{1}{3}$; $[C, (C, T)]$ is a Nash equilibrium.

If $< \frac{1}{3}$; $[C, (C, T)]$ is not a Nash equilibrium.

Is there another pure strategy Nash equilibrium?
The other possibility is when Player 1 plays T;

Again it is clear that player 2’s BT to T is (T, C):

\[ T, (T, C) \]: What conditions on \( \pi \) must be satisfied for this to be a NE?

Similar calculation:

if \( \pi \geq \frac{2}{3} \); \([T, (T, C)]\) is a NE

if \( \pi < \frac{2}{3} \); \([T, (T, C)]\) is not a NE
So:

If \( \frac{2}{3} \leq \pi \); there are two NE in pure strategies: \([C, (C, T)]\) and \([T, (T, C)]\)

If \( \frac{1}{3} \leq \pi \leq \frac{2}{3} \); there is one NE in pure strategies: \([C, (C, T)]\)

If \( \pi < \frac{1}{3} \); there is no NE in pure strategies. So equilibrium must be in mixed strategies.

Solve for a mixed strategy Nash equilibrium. (Hint: Player 1 chooses a prob. of C; and each type of player 2 chooses a prob. of C; one type of player 1 might choose a pure action)
The 1st Price Sealed-Bid Auction

Single object up for auction.

$n$ bidders

Independent private valuations for the object; Each player has a valuation independently drawn from a known distribution $v_i \in [0,1]$

Player $i$’s valuation does not depend upon any other bidder’s valuation or information

Valuations are independent and distributed with density $f(v)$ and cdf $F(v)$. 
Uniform Example:

\[ f(v) = \begin{cases} 
1 & v \in [0,1] \\
0 & \text{otherwise}
\end{cases} \]

\[ F(v) = \begin{cases} 
0 & v < 0 \\
v & v \in [0,1] \\
1 & v > 1
\end{cases} \]

Highest bid wins object, and pays bid.

Payoffs:

\[ \pi_i = \begin{cases} 
 v_i - b_i & \text{if } b_i = \max_j b_j \\
0 & \text{otherwise}
\end{cases} \]

Here, for simplicity we do not consider ties.

Different rules on how to split the ties can be considered; however, the result is not dramatically different.
Each bidder is risk neutral.

\[ \pi_i(b_i; v_i) = (v_i - b_i) \text{ Prob} \left[ b_i > \max_{j \neq i} b_j \right] \]

Since the bids are independent and the values are independent,

\[ \pi_i(b_i; v_i) = (v_i - b_i) \prod_{j \neq i} \text{ Prob}[b_i > b_j] \]
Suppose each bidder employs a bidding function $b(v)$ that is strictly increasing.

More specifically suppose that all other players bid proportional to their valuation $b_j(v_j) = \beta v_j$ and let’s find out what is the BR of player $i$ to this bidding behavior:

$$\text{Prob}[b_i > b_j] = \text{Prob}[b_i > \beta v_j] = \text{Prob}\left[v_j < \frac{b_i}{\beta}\right]$$

If $v_j$ comes from a uniform distribution:

$$\text{Prob}\left[v_j < \frac{b_i}{\beta}\right] = \frac{b_i}{\beta}$$

$$\pi_i(b_i; v_i) = (v_i - b_i) \left(\frac{b_i}{\beta}\right)^{n-1}$$
\( b_i \) chosen by \( i \) to maximize \( \pi_i(b_i; v_i) \):

\[
\frac{\partial \pi_i}{\partial b_i} = -\left(\frac{b_i}{\beta}\right)^{n-1} + \frac{n-1}{\beta} (v_i - b_i) \left(\frac{b_i}{\beta}\right)^{n-2} = 0
\]

Then

\[
\hat{b}_i = \frac{n-1}{n} v_i
\]

So if \( i \) expects everyone else to bid a fraction \( \beta \) of their valuation, it is optimal for \( i \) to bid a fraction \( \frac{n-1}{n} \) of his valuation.

So it is an equilibrium, if everyone bids a fraction \( \frac{n-1}{n} \) of their valuation.
Suppose that there are two bidders:

What is the expected value of $v_2$ conditional on $v_2 \leq v_1$?

\[
\begin{align*}
\text{Exp}(v_2 | v_2 \leq v_1) &= \frac{\int_0^{v_1} v f(v) dv}{F(v_1)} \\
&= \frac{v^2}{2} \bigg|_{v_1}^{v_1} \frac{1}{v_1} = \frac{v_1}{2}
\end{align*}
\]

Recall, the optimal bid for player 1 is $\frac{v_1}{2}$

We see that player 1 is bidding the expected value of $v_2$ conditional on $v_2 \leq v_1$
More generally, it can be shown that if there are \( n \) bidders:

\[
\hat{b}_i(v_i) = \text{Exp}\left(\max_j v_j | v_j \leq v_i\right)
\]

Remember in the 2\textsuperscript{nd} price auction, the winner pays

\[
\text{Exp}\left(\max_j v_j | v_j \leq v_i\right)
\]

So the expected payments of any bidder in the event he wins are the same in both auctions.
Revenue Equivalence Theorem:

Both 1st and 2nd price auctions yield the same expected revenues if

a) buyers are risk neutral

b) buyers are symmetric

c) valuations are independently drawn from any distribution
• **Common Values**

In many situations, player $i$ directly cares about $j$’s private information.

Examples:
- Auctions: common value auctions (Bidding for oil fields)
- Voting in committees or elections
  - Private values: e.g. political opinions
  - Common values: e.g. competence

In common value case, your information can cause me to change my mind.

**Key idea:** in common value situations, a player should condition upon being pivotal
Common Value Auctions

Two bidders

The 1\textsuperscript{st} price sealed bid auction

Suppose that player $i$ has an estimate $t_i$ of the value of the oil field

What value should player $i$ use in formulating his bid?

If every player bids $t_i$; then when a player wins, she will be subject to the \textit{winner's curse}. 
Both players’ estimates are equally reliable:

\[ v_1(t_1, t_2) = \frac{t_1 + t_2}{2} \]

If player 1 wins, this means that \( t_2 < t_1 \).

So the expected value conditional on winning is

\[ v_1(t_1 \& \text{win}) = \frac{t_1}{2} + \frac{\text{Exp}(t_2|t_2 < t_1)}{2} < t_1 \]

Suppose \( t_1 \) and \( t_2 \) are uniform on \([0; 1]\) and are independent random variables

\[ v_1(t_1 \& \text{win}) = \frac{t_1}{2} + \frac{t_1}{4} = \frac{3t_1}{4} \]
• General model

\[ v_i(t_i, t_j) = \alpha t_i + \gamma t_j \]

Private value case \( \gamma = 0 \)
Perfect common values: \( \alpha = \gamma \)

\( t_1 \) and \( t_2 \) are independent and uniformly distributed on \([0,1]\)

1\textsuperscript{st} price common value auction

Suppose that each bidder bids \( \beta t_i \)
Solve for a symmetric equilibrium where \( \beta \) is the same for the two bidders
What is player 1’s optimal bid as function of \( t_1 \); given my rival’s bidding behavior

\[
u_1(b, t_1) = \text{Prob}(b_2 < b)[\text{Exp}(v_1|t_1, b_2 < b) - b]
\]

\[
u_1(b, t_1) = \text{Prob}(\beta t_2 < b)[\text{Exp}(v_1|t_1, t_2, \beta t_2 < b) - b]
\]

\[
u_1(b, t_1) = \text{Prob}\left(t_2 < \frac{b}{\beta}\right)\left[\alpha t_1 + \gamma \text{Exp}\left(t_2 \left| t_2 < \frac{b}{\beta}\right.\right) - b\right]
\]

\[
u_1(b, t_1) = \frac{b}{\beta}\left[\alpha t_1 + \gamma \frac{b}{2\beta} - b\right]
\]
\[ u_1(b, t_1) = \frac{\alpha t_1}{\beta} b + \frac{\gamma}{2\beta^2} b^2 - \frac{1}{\beta} b^2 \]

Differentiating with respect to \( b \):

\[ \frac{\alpha t_1}{\beta} + \frac{\gamma}{\beta^2} b - \frac{2}{\beta} b = 0 \]

That is, player 1 bids a constant multiplied by his signal

In a symmetric equilibrium, \( b = \beta t_1 \); so

\[ \beta = \frac{\alpha + \gamma}{2} \]
2nd price sealed bid auction:

Claim: Bayesian Nash equilibrium where each player bids \((\alpha + \gamma)t_i\)

Suppose that player 2 is bidding \((\alpha + \gamma)t_2\)

If player 1 bids \((\alpha + \gamma)t_1\); she wins when \(t_2 < t_1\) and my payoff is

\[
(\alpha t_1 + \gamma t_2) - (\alpha + \gamma)t_2 = \alpha(t_1 - t_2) > 0
\]
If she reduces her bid, then she does not change her payoff in the event that she wins, but loses for some values of $t_2$ where $\alpha(t_1 - t_2) > 0$

So not advantageous to reduce her bid

If player 1 bids more than $(\alpha + \gamma)t_1$; then she wins for some values of $t_2$ such that $t_2 > t_1$;

But then her payoff in this case $\alpha(t_1 - t_2) < 0$

So it is a Bayesian Nash equilibrium for both players to bid $(\alpha + \gamma)t_i$
• Juries

What are the effects of different rules for juries?

a) Unanimity required for conviction (e.g. criminal trials in UK or US)
b) Only simple majority required (e.g. civil trials in UK)

Model jury decision making as a Bayesian game:

\[ \Omega = \{ \text{Guilty, Innocent} \} \]

Each juror has prior probability \( \pi \) that defendant is guilty.

Observes signal (evidence) of state
Each juror’s payoff

### Payoff Matrix

<table>
<thead>
<tr>
<th></th>
<th>Convict (Guilty)</th>
<th>Acquit (Innocent)</th>
</tr>
</thead>
<tbody>
<tr>
<td>State G</td>
<td>0</td>
<td>(-(1 - z))</td>
</tr>
<tr>
<td>State I</td>
<td>(-z)</td>
<td>0</td>
</tr>
</tbody>
</table>

### Probability Matrix

<table>
<thead>
<tr>
<th></th>
<th>g</th>
<th>i</th>
</tr>
</thead>
<tbody>
<tr>
<td>State G</td>
<td>p</td>
<td>1 - p</td>
</tr>
<tr>
<td>State I</td>
<td>1 - q</td>
<td>q</td>
</tr>
</tbody>
</table>

\(p, q > \frac{1}{2}\)
Suppose \( \mu \) is the belief of the juror that the defendant is guilty given her signal; Then:

\[
\begin{align*}
    u(\text{Convict}) &= \mu \times 0 - (1 - \mu)z \\
    u(\text{Acquit}) &= -\mu(1 - z) + (1 - \mu) \times 0
\end{align*}
\]

Juror votes to convict if and only if: \( z \leq \mu \)

And votes for acquittal if and only if: \( z \geq \mu \)

In other words acquittal is preferred if and only if

\[
\text{Prob}(G|\text{juror'}s \text{ information}) \leq z
\]
One Juror:

Suppose that the juror receives the signal $i$; Beliefs $\mu$ are formed using Bayes rule:

$$\text{Prob}(G|i) = \frac{\text{Prob}(i|G).\text{Prob}(G)}{\text{Prob}(i|G).\text{Prob}(G) + \text{Prob}(i|I).\text{Prob}(I)}$$

$$\mu(G|i) = \frac{(1 - p)\pi}{(1 - p)\pi + q(1 - \pi)}$$

Juror votes to acquit if and only if:

$$z \geq \frac{(1 - p)\pi}{(1 - p)\pi + q(1 - \pi)}$$
A similar argument for the guilty signal; since $\mu(G|i) < \mu(G|g)$ then one of the three outcomes might happen:

a) juror acquits for both signals
b) juror convicts for both signals
c) juror acts optimally: convicts for $g$; acquits if $i$ if

\[
\frac{(1 - p)\pi}{(1 - p)\pi + q(1 - \pi)} \leq z \leq \frac{p\pi}{p\pi + (1 - q)(1 - \pi)}
\]
Two Jurors:

Suppose juror 2 acts optimally (votes based on her signal); what would be the best response of juror 1:

Consider type $i$ of juror 1

If the other juror has received $i$, juror 1’s vote has no effect

Then her vote is pivotal only if juror 2’s signal is $g$

Similar to the previous case we need to construct 1’s belief:
\[
\text{Prob}(G|i, g) = \frac{\text{Prob}(i, g|G) \cdot \text{Prob}(G)}{\text{Prob}(i, g|G) \cdot \text{Prob}(G) + \text{Prob}(i, g|I) \cdot \text{Prob}(I)}
\]

\[
\mu(G|i, g) = \frac{(1 - p)p\pi}{(1 - p)p\pi + q(1 - q)(1 - \pi)}
\]

Juror 1 votes to acquit if and only if:

\[
z \geq \frac{(1 - p)p\pi}{(1 - p)p\pi + q(1 - q)(1 - \pi)}
\]
A similar argument for the guilty signal \( \mu(G|g, g) \) and the result is that there is a BNE where each player acts according to her signal if and only if:

\[
\frac{(1 - p)p\pi}{(1 - p)p\pi + q(1 - q)(1 - \pi)} \leq z \leq \frac{p^2\pi}{p^2\pi + (1 - q)^2(1 - \pi)}
\]

Comparing the lower bound on \( z \) in this case with the case of one juror, one can see that this boundary in this case is higher than with one juror.

\[
\frac{(1 - p)\pi}{(1 - p)\pi + q(1 - \pi)} < \frac{(1 - p)p\pi}{(1 - p)p\pi + q(1 - q)(1 - \pi)}
\]

Intuitively every player is less worried about convicting an innocent defendant.
Many Jurors:

Idea of many juror system with unanimity (increase the standard required for guilty verdict)

Consider a case with $n$ jurors.

Suppose that $n - 1$ other jurors vote: convict if $g$; acquit if $i$

Is it optimal for the $n^{th}$ juror to also do so?

$n^{th}$ juror knows that his vote does not matter if the information of other jurors is not $(g \cdots g)$
The $n^{th}$ juror’s vote only matters if signals for others are $(g \ldots g)$

\[ \mu(G|i, g \ldots g) = \frac{(1-p)p^{n-1}\pi}{(1-p)p^{n-1}\pi + q(1-q)^{n-1}(1-\pi)} \]
\[ = \frac{1}{1 + \frac{q}{1-p} \left(\frac{1-q}{p}\right)^{n-1} \frac{1-\pi}{\pi}} \]

So if the other $n-1$ jurors vote with their signal, the lower boundary on $z$ for the third juror will be quite close to 1. She will find it optimal to vote almost always for conviction.

There is no NE in which every player votes according to her signal.
• Problem of envelopes

Speculative trade between rational agents

Trade based on superior or differential information

No trade theorem:

   Impossibility of speculative trade when there is common knowledge that both traders are rational