• Existence of Nash Equilibria

Before we can prove the existence, we need to remind you of the fixed point theorem:

**Kakutani’s Fixed Point Theorem:** Consider $X \subset \mathbb{R}^n$ a compact convex set and a function $f: X \rightarrow X$ which has the following two properties:

i) $\forall x \in X \Rightarrow f(x)$ is nonempty and convex

ii) The graph of $f$ is closed.

Then $\exists x^* \in X : f(x^*) = x^*$. 
**Proposition:** A pure NE exist in a normal form game if for all $i \in N$:

a) The set of strategies for each player $A_i$ is a nonempty, convex, and compact subset of some Euclidean space $\mathbb{R}^M$, and  

b) $u_i(a_1, \ldots, a_N)$ is continuous in $(a_1, \ldots, a_N)$ and quasiconcave in $A_i$.

**Proof:** Let $BR_i(.)$ be the best response function for player $i$, define 

$$BR : A \rightarrow A \quad \text{and} \quad BR(a) = \prod_{i} BR_i(a_{-i})$$

1) \( \forall i \in N : u_i(.) \) is continuous & $A_i$ is compact \( \Rightarrow BR_i(a_{-i}) \) is nonempty,  

2) \( \forall i \in N : u_i(.) \) is quasiconcave in $A_i \Rightarrow BR$ is convex, and  

3) \( \forall i \in N : u_i(.) \) is continuous \( \Rightarrow BR \) has a closed graph;  

Then $BR$ has a fixed point ($\exists a^* \in A \quad \therefore \quad BR(a^*) = a^*$) which is NE.
**Proposition:** Any strategic form game with vN-M preferences in which every player has finitely many strategies has a mixed strategy NE.

**Proof:** Let $A_i$ be the set of pure strategies (with $k_i$ elements) for player $i$, define $\Delta(A_i)$ as the set of mixed strategies with probability vector $(p_1, \ldots, p_{k_i})$ for which $p_j \geq 0$ and $\sum_{j=1}^{k_i} p_j = 1$:

1) $\Delta(A_i)$ is nonempty, convex, and compact,
2) Since expected payoff is linear in the probabilities for each player, then the payoff of players over the mixed strategies are both quasiconcave in $A_i$ and continuous.

Then the problem is reduced to the last proposition.

Note: Read through Os 4.5.
• Reporting a Crime (Os 4.8)

Kitty Genovese was 28 years old and lived in New York City.

She was killed, in three separate attacks in three separate places, over a period of half an hour by a man she had never met.

The attack was witnessed by 38 witnesses.

37 of them have done nothing.

The 38th witness reluctantly called the police, she was already dead; the police arrived within two minutes.

Experiments in social psychology: lone witness to a problem is very likely to help

As the number of witnesses increases, decline in probability that the crime will be reported (at least one person helps)

Explanations in social psychology:

a) Diffusion of responsibility: more witnesses, lower the psychological cost of not helping

b) Audience inhibition: more witnesses; greater the embarrassment suffered by the helper in the event that help is inappropriate.

c) Social influence: a person infers if others are helping, then if no one else helps, it may be not appropriate to help.
Explanations assume that the benefit from helping is decreasing when the number of witnesses increases.

Is there an explanation where benefits and costs constant, but prob. that no one helps shows positive correlation with the number of witnesses?

Let’s formulate the situation as a strategic game:
Suppose there are $n$ witnesses to the crime

Witnesses should choose simultaneously between reporting ($R$) and not reporting ($N$)

Each of $n$ witnesses prefers that a crime is reported than not, then

the utility of the crime being reported is $\nu$,

the cost to the person who makes the report is $c$

and $c < \nu$

Then if $n = 1$, she will report the crime for sure ($\nu - c > 0$).
There is a pure strategy equilibrium where exactly one witness reports and no one else does.

\[ \forall j \neq i \quad a_j = N \quad \& \quad a_i = R \]

\[ \forall j \neq i \quad u_j(a_j = N, a_i = R) = v > v - c \]

\[ \& \quad u_i(a_i = R, a_{j\neq i} = N) = v - c > 0 \]

No pure strategy equilibrium where more than one person reports.

How to coordinate? How will we realize that \( i \) is the person who will report?
Is there a symmetric mixed strategy equilibrium each person reports with probability $p$:

$$\Pr(\text{no one else reports}) = (1 - p)^{n-1}$$

$$u(N, n, p) = \Pr(\text{at least one other player reports}) \times v = (1 - (1 - p)^{n-1})v$$

$$u(R, n, p) = v - c$$
In the mixed NE each player is indifferent between $N$ and $R$:

$$\Pr(\text{at least one other player reports}) \times v = v - c$$

RHS is independent of $n$; so LHS must also be same for all values of $n$:

$$\Pr(\text{at least one other player reports}) = \frac{v - c}{v}$$

$$\Pr(\text{no one reports}) = \Pr(\text{at least one other player reports}) \times (1 - p)$$

So if $p$ is decreasing in $n$; $\Pr(\text{no one reports})$ increases in $n$. 
The probability of each person reports in the symmetric mixed strategy NE is:

\[(1 - (1 - p)^{n-1})v = v - c\]

Then

\[p = 1 - \left(\frac{c}{v}\right)^{\frac{1}{n-1}}\]

Numerical example (suppose \(c = \frac{v}{3}\)):

<table>
<thead>
<tr>
<th>(n)</th>
<th>(p)</th>
<th>Pr(the crime will not be reported)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>66.7%</td>
<td>11.1%</td>
</tr>
<tr>
<td>3</td>
<td>42.3%</td>
<td>19.2%</td>
</tr>
<tr>
<td>5</td>
<td>24.0%</td>
<td>25.3%</td>
</tr>
<tr>
<td>10</td>
<td>11.5%</td>
<td>29.5%</td>
</tr>
<tr>
<td>38</td>
<td>2.9%</td>
<td>32.4%</td>
</tr>
</tbody>
</table>
• **Strictly Competitive Games**

**Definition:** A two-player strategic game is a *strictly competitive game* if for any two strategy profiles $a, b \in A$ we have:

$$u_1(a) \geq u_1(b) \iff u_2(a) \leq u_2(b).$$

In particular if $\forall a \in A \; \therefore u_1(a) + u_2(a) = 0$ then the game is a *zero-sum game*.

We can define the strictly competitive games for games with vN-M preferences in a similar fashion; in this case the comparisons can be done for mixed strategies.

The theoretical importance of strictly competitive game is that something can be said about the qualitative characteristics of Equilibria.
**Definition:** In a two-player strictly competitive game with vN-M preferences, a *max-minimizing mixed strategy* for player $i$ is the solution to:

$$\max_{\alpha_i} \min_{\alpha_{-i}} U_i(\alpha_i, \alpha_{-i})$$

If $\alpha_i^*$ is player $i$’s max-minimizing strategy then $\min_{\alpha_{-i}} U_i(\alpha_i^*, \alpha_{-i})$ is her *max-minimized payoff*.

A player’s max-minimizing mixed strategy guarantees her max-minimized payoff. Furthermore, no strategy guarantees her a higher payoff.
Suppose $\alpha^*$ is a NE for a strictly competitive game, then
\[
U_i(\alpha_i^*, \alpha_{-i}^*) \geq U_i(\alpha_i, \alpha_{-i}^*) \quad \text{for } \forall \alpha_i
\]

We also have
\[
U_i(\alpha_i, \alpha_{-i}^*) \geq \min_{\alpha_{-i}} U_i(\alpha_i, \alpha_{-i}) \quad \text{for } \forall \alpha_i
\]

So
\[
U_i(\alpha_i^*, \alpha_{-i}^*) \geq \min_{\alpha_{-i}} U_i(\alpha_i, \alpha_{-i}) \quad \text{for } \forall \alpha_i
\]
And it is trivial that

\[ U_i(\alpha_i^*, \alpha_{-i}^*) \geq \max_{\alpha_i} \min_{\alpha_{-i}} U_i(\alpha_i, \alpha_{-i}) \]

Then we proved this Lemma:

**Lemma:** The payoff of each player in a mixed strategy NE of a strategic game is at least equal to her max-minimized payoff.

We now want to go one step further and prove that if a strictly competitive game has a NE, it is the same as its max-minimizing strategy.
Proposition: Consider a strictly competitive game with vN-M preferences:

i) If strategy profile $\alpha^*$ is a mixed strategy NE, then $\alpha_i^*$ is a max-minimizer strategy for player $i$:

$$\max_{\alpha_1} \min_{\alpha_2} U_1(\alpha_1, \alpha_2) = \min_{\alpha_2} \max_{\alpha_1} U_1(\alpha_1, \alpha_2) = U_1(\alpha_1^*, \alpha_2^*)$$

ii) If

$$\max_{\alpha_1} \min_{\alpha_2} U_1(\alpha_1, \alpha_2) = \min_{\alpha_2} \max_{\alpha_1} U_1(\alpha_1, \alpha_2),$$

$\alpha_1^*$ is a max-minimizer strategy for player 1, and $\alpha_2^*$ is a max-minimizer strategy for player 2, Then $(\alpha_1^*, \alpha_2^*)$ is a mixed strategy NE of the game.
• Empirical Evidence and Equilibria

Do players, in real life, play [mixed strategies] Nash equilibrium?

There are some papers investigate the players’ behaviour in playing the NE in laboratory experiments; for example:


Found behavior quite far from predictions of equilibrium.

Argued that has happened because inexperience and lack of incentives.

Concludes that his experimental evidence is consistent with mixed NE.


Rejects O’Neill’s findings.
Another example Biel, “Equilibrium Play and Best Response to (Stated) Beliefs in Constant Sum Games”, *Games & Econ. Behavior* (2009)

There are limitations on laboratory studies:

1) The behavior in the simplified, artificial setting of games played in such laboratories need not represent real-life behavior.

2) Even if individuals behave in ways that are inconsistent with optimizing behavior in the laboratory studies, market forces may force such behavior in the real world.

3) Interpretation of laboratory experiments rely on the assumption that the subjects are maximizing the monetary outcome of the game (which even in lab experiments we are not sure that are high enough to encourage players), whereas there may be other preferences present among players in real life that might change the results.
Here we focus on some new papers using non-experimental data.

Walker and Wooders, “Minmax play at Wimbledon”, *AER* (2001)

In a game of tennis, server seeks to maximize probability of winning this point.

Server can serve L or R;

Receiver can anticipate L or R;

Each strategy combination (L, L); (L, R); (R, L); (R, R) determines a probability of winning the point.

2 × 2 game with unique Nash equilibrium in mixed strategies.
This is a zero sum game, and Nash equilibrium strategies are also known as minmax strategies.

Point game (payoff matrix) varies according who is serving, and also between deuce court and ad court.

Assume that this payoff matrix does not change otherwise

i.e. every game where Federer is serving to Nadal on ad court is the same over the entire match.

We therefore observe repeated plays of the same game over the match.
Observe direction of serve (L or R)

Do not observe direction of anticipation by receiver.

Fix one of the 4 point games in the match.

\[ \Pr(\text{Win} \mid \text{serves L}) = \Pr(\text{Win} \mid \text{serves R}) = p \]

Outcomes when serve L are independent draws from a Bernoulli trial with success probability \( \Pr(W \mid L) \)

Outcomes when serve R are independent draws from a Bernoulli trial with success probability \( \Pr(W \mid R) = \Pr(W \mid L) \)
1st Testable Prediction:

Statistical test whether actual winning frequencies are significantly different from each other.

In any match, 4 different point games

10 grand slam matches - 40 different point games.

Since \( p \) is not known the Pearson test statistic is distributed as \( \chi^2 \) with one degree of freedom.

If we test in 40 games and reject at 5% level, we should expect 2 rejections given that the null (theory) is true.

one rejection (Sampras v Agassi, 1995)
2\textsuperscript{nd} Testable Prediction:

Server's choices must be serially independent in each point game.

Look at the number of runs

e.g. in 6 element sequence LRRRLLL there are 3 runs.

If there are \( n \) serves, look at the number of runs \( k \)

If \( k \) is too large - too many changes of direction in serve - negative correlation.

If \( k \) is too small - runs are very long - too few changes of direction, positive correlation.
Rejection of serial independence in 5 out of 40 point games.

Overall, too many changes of direction relative to independence.

Psychological literature - subjects have difficulty generating random sequences.

Belief in law of small numbers

Reasonably good support for predictions of mixed equilibrium in tennis serve.
Penalty Kicks in Soccer


