Neoclassical Growth Model

Seyed Ali Madanizadeh
Sharif U. of Tech.

march 26,2017

The development of growth theory

- **Smith (1776), Malthus (1798), Ricardo (1817), Marx (1867)**
  growth falls in the presence of a fixed factor
- **Harrod (1939) and Domar (1946)**
  models with little factor substitution and an exogenous saving rate
- **Solow (1956) and Swan (1956)**
  factor substitution, an exogenous saving
- **Ramsey (1928), Cass (1965) and Koopmans (1965)**
  growth with consumer optimisation (intertemporal substitution)

Ramsey vs Solow

- *Solow: agents (or the dictator) follow a simplistic linear rule for consumption and investment.* *In spite of being limited and inappropriate to account for the growth dynamics of modern economies, the disparities of economic growth across time and space, the Solow model is the starting point for almost all analysis of economic growth.*
- *Ramsey:agents (or the dictator) choose consumption and investment optimally so as to maximize their individual utility (or social welfare.)* *Establishes the benchmark model for modern dynamic macroeconomics and optimal intertemporal allocation of resources.*

The Neoclassical Growth Model

The Solow growth model is predicated on a constant saving rate.

Instead, it would be much more satisfactory to specify the preference orderings of individuals, as in standard general equilibrium theory, and derive their decisions from these preferences.

This will enable us both to have a better understanding of the factors that affect savings decisions and also to discuss the optimality of equilibriain other words, to pose and answer questions related to whether the (competitive) equilibria of growth models can be improved upon.

The notion of improvement here will be based on the standard concept of Pareto optimality, which asks whether some households can be made better off without others being made worse off.

Preferences, Technology and Demographics

- Consider an infinite-horizon economy in continuous time. We assume that the economy admits a representative household with instantaneous utility function: $u(c(t))$

  $u(c)$ is a strictly increasing, concave, twice continuously differentiable with derivatives u' and u'', and satisfies the following Inada type assumptions:

$$\lim_{c \to 0} u'(c) = \infty, \lim_{c \to \infty} u'(c) = 0, \tag{1}$$

  Population within each household grows at the rate n, starting with $L(0) = 1$, so that total population is:

$$L(t) = exp(nt) \tag{2}$$

  All members of the household supply their labor inelastically.

Preferences, Technology and Demographics

- Our baseline assumption is that the household is fully altruistic towards all of its future members, and always makes the allocations of consumption cooperatively.

  This implies that the objective function of each household at time t = 0, U (0), can be written as:

$$U(0) \equiv \int_0^\infty exp(-(\rho - n)t)u(c(t))dt, \tag{3}$$

  ρ is the subjective discount rate, and the effective discount rate is $\rho - n$ and $c(t) \equiv \frac{C(t)}{L(t)}$ where C (t) is total consumption and L(t) is the size of the representative household (equal to total population, since the measure of households is normalized to 1). This implies that a total utility of $L(t)u(c(t)) = exp(nt)u(c(t))$.

  utility at time t is discounted back to time 0 with a discount rate of $exp(-\rho t)$.

Preferences, Technology and Demographics

- We assume throughout that $\rho > n$. it ensures that there is discounting of future utility streams and makes sure that in the model without growth, discounted utility is finite.

  We have no technological progress. markets are competitive, and the production possibilities set of the economy is represented by the aggregate production function:

$$Y(t) = F(K(t), L(t)) \tag{4}$$

  The constant returns to scale feature enables us to work with the per capita production function f () such that, output per capita is given by

$$y(t) = \frac{Y(t)}{L(t)} = F[\frac{(K(t)}{L(t)}, 1] \tag{5}$$

$$\equiv f(k(t)), where \quad k(t) \equiv \frac{K(t)}{L(t)} \tag{6}$$

Preferences, Technology and Demographics

- Competitive factor markets then imply that, at all points in time, the rental rate of capital and the wage rate are given by:

$$R(t) = F_K[K(t), L(t)] = f'(k(t)) \quad (7)$$

$$w(t) = F_L[K(t), L(t)] = f(k(t)) - k(t)f'(k(t)) \quad (8)$$

we denote the asset holdings of the representative household at time t by A(t). Then we have the following law of motion for the total assets of the household:

$$\dot{A}(t) = r(t).A(t) + w(t)L(t) - c(t)L(t) \quad (9)$$

r(t) is the risk-free market flow rate of return on assets, and w(t)L(t) is the flow of labor income earnings of the household. Defining per capita assets as $a(t) \equiv \frac{A(t)}{L(t)}$

$$\dot{a}(t) = (r(t) - n)a(t) + w(t) - c(t) \quad (10)$$

Preferences, Technology and Demographics

- assets per capita will be equal to the capital stock per capita (or the capital-labor ratio in the economy), that is, $a(t) = k(t)$.
  since there is no uncertainty here and a depreciation rate of , the market rate of return on assets will be given by:

$$r(t) = R(t) - \delta \tag{11}$$

  The equation 10 is only a flow constraint. As already noted above, it is not sufficient as a proper budget constraint on the individual.
  To see this, let us write the single budget constraint of the form:

$$\int_0^T c(t)L(t)exp(\int_0^T r(s)ds)dt + A(T)$$

$$= \int_0^T w(t)L(t)exp(\int_t^T r(s)ds)dt + A(0)exp(\int_0^T r(s)ds) \tag{12}$$

  Differentiating eq. 12 with respect to T and dividing $L(t)$ gives eq. 10.

Preferences, Technology and Demographics

- In the infinite-horizon case, we need a similar boundary condition. This is generally referred to as the no-Ponzi-game condition, and takes the form:

$$\lim_{t \to \infty} a(t) exp(-\int_0^t (r(s) - n) ds) \geq 0. \tag{13}$$

This condition is stated as an inequality, to ensure that the individual does not asymptotically tend to a negative wealth.

the transversality condition ensures that the individual would never want to have positive wealth asymptotically, so the noPonzi-game condition can be alternatively stated as: 14

$$\lim_{t \to \infty} a(t) exp(-\int_0^t (r(s) - n) ds) = 0. \tag{14}$$

In what follows we will use eq. 13, and then derive eq. 14 using the transversality condition explicitly.

- To understand where this form of the no-Ponzi-game condition comes from, multiply both sides of eq. 12 by $exp(-\int_0^T r(s)ds)$ to obtain:

$$\int_0^T c(t)L(t)exp(-\int_0^t r(s)ds)dt + exp(-\int_0^T r(s)ds).A(T)$$

$$= \int_0^T w(t)L(t)exp(-\int_0^t r(s)ds)dt + A(0)$$

then divide everything by $L(0)$ and $L(t)$ grows at the rate n:

$$\int_0^T c(t)exp(-\int_0^t (r(s)-n)ds)dt + exp(-\int_0^T (r(s)-n)ds)a(T)$$

$$= \int_0^T w(t)exp(-\int_0^t (r(s)-n)ds)dt + a(0)$$

Preferences, Technology and Demographics

- Now take the limit as $T \to \infty$ and use the no-Ponzi-game condition 14 to obtain:

$$\int_0^\infty c(t) exp(-\int_0^t (r(s) - n) ds) dt$$

$$= a(0) + \int_0^\infty w(t) exp((-\int_0^t (r(s) - n) ds) dt$$

which requires the discounted sum of expenditures to be equal to initial income plus the discounted sum of labor income.

This derivation makes it clear that the no-Ponzi-game condition eq. 14 essentially ensures that the individuals lifetime budget constraint holds in infinite horizon.

Characterization of Equilibrium

- **Definition 1:**A competitive equilibrium of the Ramsey economy consists of paths of *consumption, capital stock, wage rates* and *rental rates of capital*, $[C(t), K(t), W(t), R(t)$, such that the representative household maximizes its utility given initial capital stock K(0) and the time path of prices [w(t), R(t)] and all markets clear.

  Definition 2: A competitive equilibrium of the Ramsey economy consists of paths of per capita consumption, capital-labor ratio, wage rates and rental rates of capital, $[c(t), k(t), w(t), R(t)$.

Household Maximization

- we should maximize eq. 3 subject to eq. 10 and eq. 14. we first ignore eq. 14 and set up the current-value Hamiltonian:

$$\hat{H}(a, c, \mu) = u(c(t)) + \mu(t)[w(t) + (r(t) - n)a(t) - c(t)], \quad (15)$$

a, c, μ:state, control and current-value costate variable and FOCs:

$$\hat{H}_c(a, c, \mu) = u'((c(t)) - \mu(t) = 0$$

$$\hat{H}_a(a, c, \mu) = \mu(t)(r(t) - n) = -\dot{\mu}(t) + (\rho - n)\mu(t)$$

$$\lim_{t\to\infty}[exp(-(\rho - n)t)\mu(t)a(t)] = 0.$$

We can next rearrange the second condition to obtain:

$$\frac{\dot{\mu}}{\mu} = -(r(t) - \rho) \quad (16)$$

which states that the multiplier changes depending on whether the rate of return on assets is currently greater than or less than the discount rate of the household.

Household Maximization

- Next, the first necessary condition above implies that

$$u'(c(t)) = \mu(t) \tag{17}$$

differentiating this with respect to t and divide by $\mu(t)$, which yields:

$$\frac{u''(c(t))c(t)}{u'(c(t))}\frac{\dot{c}(t)}{c(t)} = \frac{\dot{\mu}}{\mu}. \tag{18}$$

- Substituting this into 16, we obtain the famous consumer Euler equation:

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))}(r(t) - \rho) \tag{19}$$

- where

$$\varepsilon_u(c(t) \equiv -\frac{u''(c(t))c(t)}{u'(c(t))} \tag{20}$$

is the elasticity of the marginal utility $u'(c(t))$ and the inverse of the intertemporal elasticity of substitution.

Household Maximization

- The intertemporal elasticity of substitution regulates the willingness of individuals to substitute consumption over time. The elasticity between the dates t and $s > t$ is defined as

$$\sigma_u(t,s) = -\frac{d\log(c(s))/c(t)}{d\log(u'(c(s))/(u'(c(t)))}. \tag{21}$$

As $s \downarrow t$, we have

$$\sigma_u(t,s) \longrightarrow \sigma_u(t) = -\frac{u'(c(t))}{u''(c(t))c(t)} = \frac{1}{\varepsilon_u(c(t))} \tag{22}$$

This is not surprising, since the concavity of the utility function determines how willing individuals are to substitute consumption over time.

Household Maximization

- Next, integrating eq. 16 , we have

$$\mu(t) = \mu(0) exp(-\int_0^t (r(t) - \rho) ds$$

$$= u'(c(0)) exp(-\int_0^t (r(t) - \rho) ds$$

- Now substituting into the transversality condition, we have

$$\lim_{t\to\infty} [exp(-(\rho - n)t) a(t) u'(c(0)) exp(-\int_0^t (r(s) - \rho) ds] = 0,$$

$$\lim_{t\to\infty} [a(t) exp(-\int_0^t (r(s) - \rho) ds] = 0,$$

which implies that the strict no-Ponzi condition, has to hold.

Household Maximization

- we can define an average interest rate between dates 0 and t as

$$\bar{r}(t) = \frac{1}{t}\int_0^t r(s)ds.$$

- and the transversality condition can be written as

$$\lim_{t\to\infty}[exp(-(\bar{r}(t) - n)t)a(t)] = 0$$

- we can integrate eq.19, to obtain

$$c(t) = c(0)exp(\int_0^t \frac{r(s) - \rho}{\varepsilon(c(s))}ds)$$

Equilibrium Prices

- the market rate of return for consumers, $r(t)$, is given by:

$$r(t) = f'(k(t)) - \delta.$$

- Substituting this into the consumers problem, we have

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))}(f'(k(t))\delta - \rho) \tag{23}$$

Optimal Growth

- The optimal growth problem, defined as the capital and consumption path chosen by a benevolent social planner trying to achieve a Pareto optimal outcome.

  recall that in an economy that admits a representative household, the optimal growth problem simply involves the maximization of the utility of the representative household subject to technology and feasibility constraints.

  That is,

$$\max_{[k(t),c(t)]t=0,\infty} \int_0^\infty exp(-(\rho - n)t)u(c(t))dt$$

  subject to

$$\dot{k}(t) = f(k(t)) - (n+\delta)k(t) - c(t), \quad and \quad k(0) > 0.$$

Optimal Growth

- the current-value Hamiltonian:

$$\dot{H}(k,c,\mu) = u(c(t)) + \mu(t)[f(k(t)) - (n+\delta)k(t) - c(t)],$$

  with state variable k, control variable c and current-value costate variable μ
- FOCs:

$$\dot{H}_c(K,c,\mu) = 0 = u'(c(t)) - \mu(t),$$

$$\dot{H}_k(K,c,\mu) = -\dot{\mu}(t) + (\rho - n)\mu(t) = \mu(t)(f'(k(t)) - \delta - n),$$

$$\lim_{t\to\infty}[exp(-(\rho - n)t)\mu(t)k(t)] = 0$$

- as before, these optimality conditions imply

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))}(f'(k(t)) - \delta - \rho),$$

Optimal Growth

- as before, these optimality conditions imply

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))}(f'(k(t)) - \delta - \rho),$$

- and the transversality condition:

$$\lim_{t\to\infty}[k(t)exp(-\int_0^t (f'(k(s)) - \delta - n)ds)] = 0,$$

- This establishes that the competitive equilibrium is a Pareto optimum and that the Pareto allocation can be decentralized as a competitive equilibrium.

Steady-State Equilibrium

- A steady-state equilibrium is defined as an equilibrium path in which capital-labor ratio, consumption and output are constant. Therefore, $\dot{c(t)} = 0$.
- This implies that as long as $f(k^*) > 0$,irrespective of the exact utility function, we must have a capital-labor ratio k^* such that

$$f'(k^*) = \rho + \delta, \tag{24}$$

- This equation pins down the steady-state capital-labor ratio only as a function of the production function, the discount rate and the depreciation rate.

Steady-State Equilibrium

- This corresponds to the modified golden rule, rather than the golden rule we saw in the Solow model.
- The modified golden rule involves a level of the capital stock that does not maximize steady-state consumption, because earlier consumption is preferred to later consumption.
- This is because of discounting, which means that the objective is not to maximize steady-state consumption, but involves giving a higher weight to earlier consumption.

Steady-State Equilibrium

- Given k^*, the steady-state consumption level is straightforward to determine as:

$$c^* = f(k^*) - (n + \delta)k^*, \tag{25}$$

- As with the basic Solow growth model, there are also a number of straightforward comparative static results that show how the steady-state values of capital-labor ratio and consumption per capita change with the underlying parameters. For this reason, let us again parameterize the production function as follows

$$f(k) = \alpha \tilde{f}(k)$$

- where $a > 0$, so that a is again a shift parameter, with greater values correspondin to greater productivity of factors. Since $f(k)$ satisfies the regularity conditions imposed above, so does $\tilde{f}(k)$.

Steady-State Equilibrium

Consider the neoclassical growth model described above, with Assumptions, and suppose that $f(k) = a\tilde{f}(k)$. Denote the steadystate level of the capital-labor ratio by $k^*(\alpha, \rho, n, \delta)$ and the steady-state level of consumption per capita by $c^*(\alpha, \rho, n, \delta)$ when the underlying parameters are α, ρ, n and δ. Then we have

$$\frac{\partial k^*(\alpha, \rho, n, \delta)}{\partial \alpha} > 0, \quad \frac{\partial k^*(\alpha, \rho, n, \delta)}{\partial \rho} < 0,$$

$$\frac{\partial k^*(\alpha, \rho, n, \delta)}{\partial n} = 0, \quad \frac{\partial k^*(\alpha, \rho, n, \delta)}{\partial \delta} < 0$$

$$\frac{\partial c^*(\alpha, \rho, n, \delta)}{\partial \alpha} > 0, \quad \frac{\partial c^*(\alpha, \rho, n, \delta)}{\partial \rho} < 0,$$

$$\frac{\partial c^*(\alpha, \rho, n, \delta)}{\partial n} < 0, \quad \frac{\partial c^*(\alpha, \rho, n, \delta)}{\partial \delta} < 0$$

Steady-State Equilibrium

- The new results here relative to the basic Solow model concern the comparative statics with respect the discount factor.
- In particular, instead of the saving rate, it is now the discount factor that affects the rate of capital accumulation.
- There is a close link between the discount rate in the neoclassical growth model and the saving rate in the Solow model.
- Loosely speaking, a lower discount rate implies greater patience and thus greater savings.

  In the model without technological progress, the steady-state saving rate can be computed as

$$s^* = \frac{\delta k^*}{f(k^*)}$$

Steady-State Equilibrium

- Another interesting result is that the rate of population growth has no impact on the steady state capital-labor ratio, which contrasts with the basic Solow model.

  This result depends on the way in which intertemporal discounting takes place.

- Another important result, which is more general, is that k^* and thus c^* do not depend on the instantaneous utility function $u(0)$.

- The form of the utility function only affects the transitional dynamics (which we will study next), but has no impact on steady states. This is because the steady state is determined by the modified golden rule.

  This result will not be true when there is technological change, however.

Transitional Dynamics

Transitional dynamics in the basic Solow model were given by a single differential equation with an initial condition. This is no longer the case, since the equilibrium is determined by two differential equations, repeated here for convenience:

$$\dot{k}(t) = f(k(t)) - (n+\delta)k(t) - c(t) \tag{26}$$

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))}(f'(k(t)) - \delta - \rho). \tag{27}$$

Moreover, we have an initial condition $k(0) > 0$, also a boundary condition at infinity, of the form

$$\lim_{t\to\infty}[k(t)exp(-\int_0^t (f'(k(s)) - \delta - n)ds)] = 0,$$

Transitional Dynamics

- The consumption level (or equivalently μ) is the control variable, and $c(0)$ (or $\mu(0)$) is free. It has to adjust so as to satisfy the transversality condition.
- Since $c(0)$ or $\mu(0)$ can jump to any value, we need that there exists a one-dimensional curve tending to the steady state.
- In fact, as in the q-theory of investment, if there were more than one paths tending to the steady state, the equilibrium would be indeterminate, since there would be multiple values of $c(0)$ that could be consistent with equilibrium.

Transitional Dynamics

- The correct notion of stability in models with state and control variables is one in which the dimension of the stable curve is the same as that of the state variables, requiring the control variables jump on to this curve.
- The economic forces are such that the correct notion of stability is guaranteed and indeed there will be a one-dimensional manifold of stable solutions tending to the unique steady state.

Transitional Dynamics

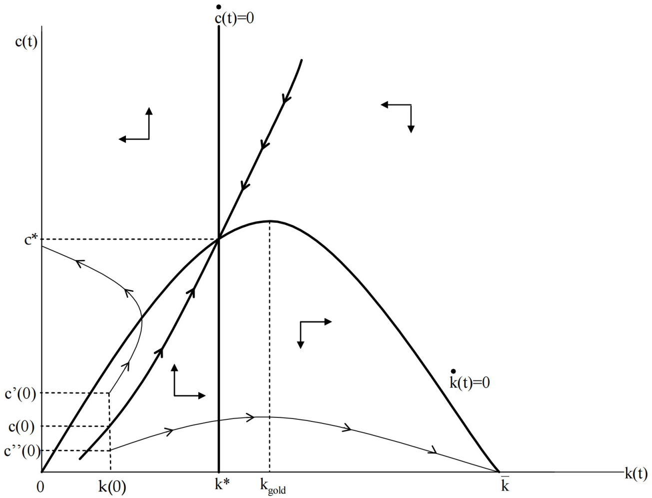

Figure: Transitional dynamics in the baseline neoclassical growth model

Transitional Dynamics

- The vertical line is the locus of points where $\dot{c} = 0$.The reason why the $\dot{c} = 0$ locus is just a vertical line is that in view of the consumer Euler equation 27 , only the unique level of k^* given by eq. 24 can keep per capita consumption constant.
- The inverse U-shaped curve is the locus of points where $\dot{k} = 0$ in 26.
- The intersection of these two loci defined the steady state.
- If the capital stock is too low, steady-state consumption is low, and if the capital stock is too high, then the steady-state consumption is again low.
- There exists a unique level, k_{gold} that maximizes the state-state consumption per capita.
- The $\dot{c} = 0$ locus intersects the $\dot{k} = 0$ locus always to the left of k_{gold}.

Transitional Dynamics

- It is clear that there exists a unique stable arm, the one-dimensional manifold tending to the steady state.
- All points away from this stable arm diverge, and eventually reach zero consumption or zero capital stock.
- If initial consumption, $c(0)$, started above this stable arm, say at $c'(0)$, the capital stock would reach 0 in finite time, while consumption would remain positive.
- Initial values of consumption above this stable arm cannot be part of the equilibrium.
- If the initial level of consumption were below it, for example, at $c''(0)$, consumption would reach zero, thus capital would accumulate continuously until the maximum level of capital (reached with zero consumption)$k > k_{gold}$.

Transitional Dynamics

In the neoclassical growth model described, with its Assumptions, there exists a unique equilibrium path starting from any $k(0) > 0$ and converging to the unique steady-state (k^*, c^*) with k^* given by eq. 24.
Moreover, if $k(0) < k^*$, then $k(t) \uparrow k^*$ and $c(t) \uparrow c^*$, whereas if $k(0) > k^*$, then $k(t) \downarrow k^*$ and $c(t) \downarrow c^*$.

Transitional Dynamics

- Recall the two differential equations determining the equilibrium path:

$$\dot{k}(t) = f(k(t)) - (n+\delta)k(t) - c(t)$$

And

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))}(f'(k(t)) - \delta - \rho).$$

- Linearizing these equations around the steady state (k^*, c^*), we have

$$\dot{k} = constant + (f'(k^*) - n - \delta)(k - k^*) - c$$

$$\dot{c} = constant + \frac{c^* f''(k^*)}{\varepsilon_u(c^*)}(k - k^*)$$