The Solow Growth Model

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Economic Growth Facts

1. In the data, real GDP/capita has been growing over time for most countries.
2. There are large differences in per capita income levels across countries: $34k vs $200.
3. In similar countries, economies with lower real GDP per capita had faster growth rates. But this is not true globally.
4. Some economies were converging to each other.
5. Still there are some stagnations.

Questions

1. What are the factors that lead to economic growth?
2. Why do some countries grow fast and others slow? (East Asians vs sub-Saharan African countries)
3. What policies increase real GDP per capita?
4. How did rich countries sustain growth rates of 2%?

Solow Model

- Production function:

$$Y_t = A_t.F\left(K_t, L_t\right)$$

- Capital accumulates through investment:

$$K_{t+1} = (1-\delta)\, K_t + I_t$$

  and

$$Y = C + I$$

  I is the investment: Purchases of capital: In the data 14% of GDP.
- F is diminishing marginal product of capital and labor.

Solow Model

F is constant returns to scale, therefore Real GDP per capita is

$$\begin{aligned} Y/L &= AF\left(K/L, L/L\right) \\ y &= AF(k,1) \\ y &= Af\left(k\right) \end{aligned}$$

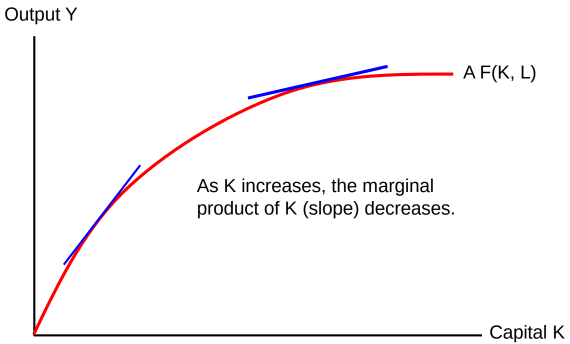
Solow Model

- Example:

$$Y = AK^{\alpha}L^{1-\alpha}$$

Therefore

$$y = Ak^{\alpha}$$



Growth Accounting

$$\begin{aligned}\log Y_t &= \log A_t + \alpha \log K_t + (1-\alpha) \log L_t \\ g_Y &= g_A + \alpha g_K + (1-\alpha) g_L\end{aligned}$$

$$\frac{\Delta Y}{Y} = \frac{\Delta A}{A} + \alpha \frac{\Delta K}{K} + (1-\alpha) \frac{\Delta L}{L}$$

or

$$\frac{\Delta y}{y} = \frac{\Delta A}{A} + \alpha \frac{\Delta k}{k}$$

Simplifying assumption

$$I_t = sY_t$$

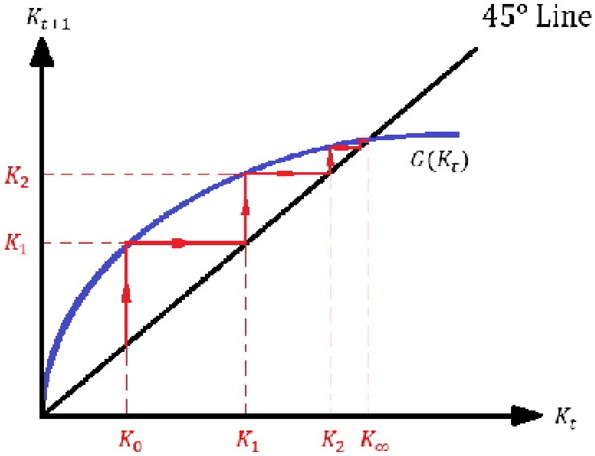
national savings rate: s

Solow Model

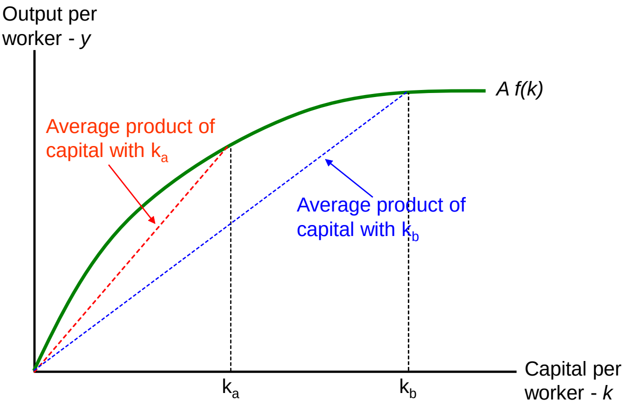
- Population growth rate: g_n
- Rewriting the law of motion:

$$
\begin{aligned}
K_{t+1} &= (1-\delta) K_t + sY_t \\
\frac{K_{t+1}}{L_{t+1}} \frac{L_{t+1}}{L_t} &= (1-\delta) \frac{K_t}{L_t} + s\frac{Y_t}{L_t} \\
k_{t+1} (1+g_n) &= (1-\delta) k_t + sy_t \\
k_{t+1} &= \frac{sy_t}{1+g_n} + \frac{1-\delta}{1+g_n} k_t \\
k_{t+1} &= \frac{sAf(k_t)}{1+g_n} + \frac{1-\delta}{1+g_n} k_t = G(k_t)
\end{aligned}
$$

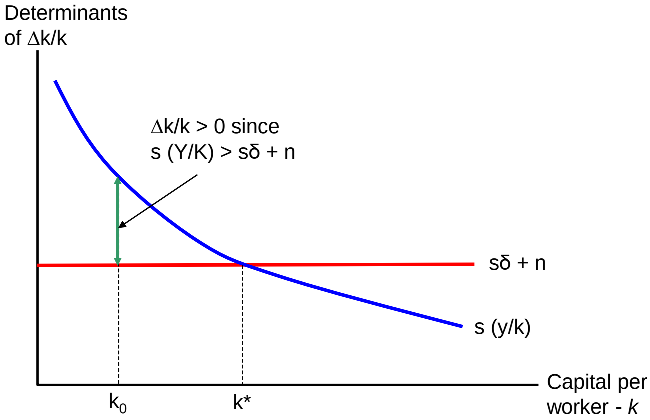
Capital choices and capital dynamics



Average product of capital



Capital growth rate



Growth variable

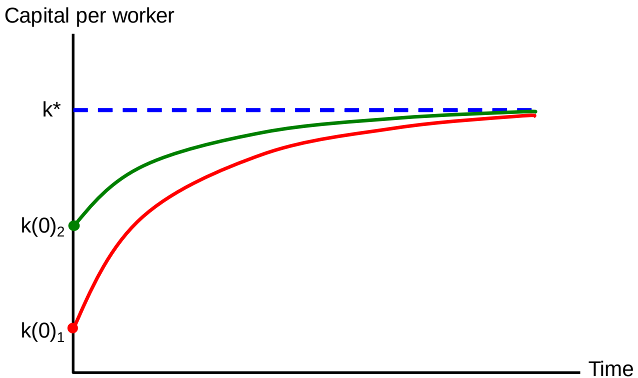
$$\frac{\Delta Y}{Y} = \frac{\Delta A}{A} + \alpha \frac{\Delta K}{K} + (1-\alpha)\frac{\Delta L}{L}$$

$$\frac{\Delta k}{k} = s\left(\frac{y}{k}\right) - \delta - n$$

$$\frac{\Delta y}{y} = \alpha\left(s\left(\frac{y}{k}\right) - \delta - n\right)$$

Predictions

- Convergence
- Data: There is conditional convergence: economies with similar characteristics converge
- Lower K, higher growth



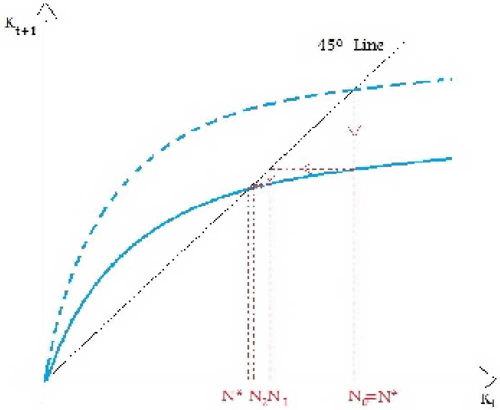
Steady state level

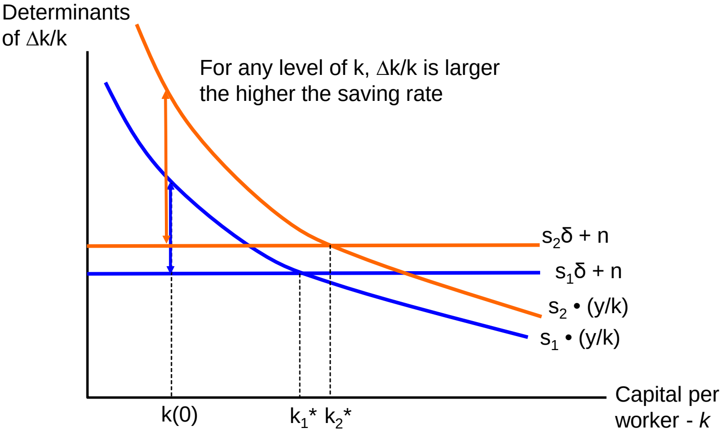
$$
\begin{aligned}
k^* &= G\left(k^*\right) = \frac{sAf\left(k^*\right)}{1+g_n} + \frac{1-\delta}{1+g_n}k^* \\
&= \frac{sAk^{*\alpha}}{1+g_n} + \frac{1-\delta}{1+g_n}k^* \\
k^* &= A^{\frac{1}{1-\alpha}}\left(\frac{s}{g_n+\delta}\right)^{\frac{1}{1-\alpha}} \\
i^* &= \left(\frac{g_n+\delta}{1+g_n}\right)k^*
\end{aligned}
$$

$$
\begin{aligned}
y^* &= A^{\frac{1}{1-\alpha}}\left(\frac{s}{g_n+\delta}\right)^{\frac{\alpha}{1-\alpha}} \\
Y_t &= A^{\frac{1}{1-\alpha}}\left(\frac{s}{g_n+\delta}\right)^{\frac{\alpha}{1-\alpha}} L_0\left(1+g_n\right)^t
\end{aligned}
$$

Comparative statics

- Changes in $s, g_n, A, \delta, \alpha$

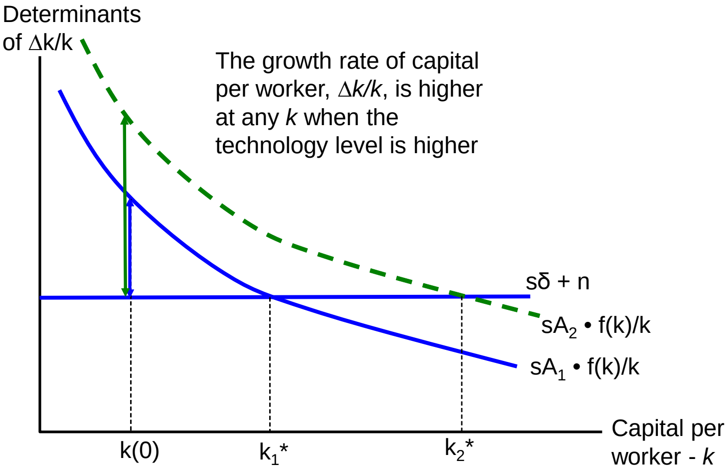



Determinants
of Δk/k
For any level of k, Δk/k is larger
the higher the saving rate
$s_2\delta + n$
$s_1\delta + n$
$s_2 \bullet (y/k)$
$s_1 \bullet (y/k)$
Capital per
worker - *k*
k(0)
k_1^* k_2^*

A Change in the Saving Rate

- Compare two countries with different savings rates. How does that affect $\Delta k/k$, k^* and y^*.
- A higher saving rate raises $\Delta k/k$, which rate remains higher during the transition to the steady state.
- In the long run, $\Delta k/k$ and $\Delta y/y$ are equal to zero for any saving rate, but a higher saving rate leads to higher steady-state k^* and y^*.

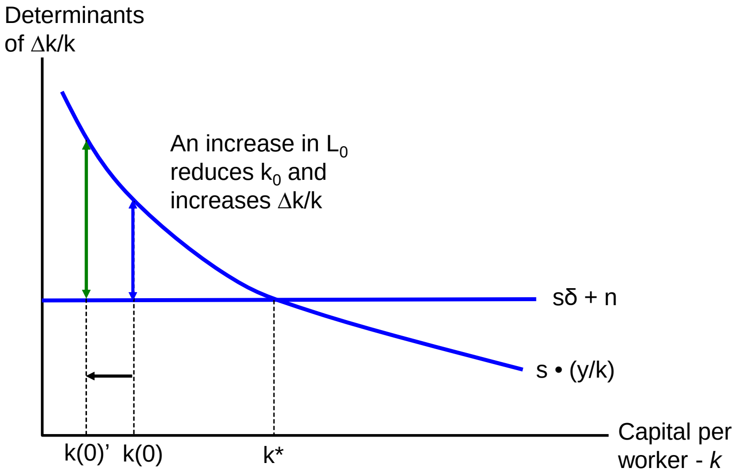
A Change in the Technology Level



A Change in the Technology Level

- What happens to $\Delta k/k$, k^* and y^* if there is a change in technology?
- In the short run, an increase in A raises $\Delta k/k$ and $\Delta y/y$, which remain higher during the transition to the steady state.
- In the long run, $\Delta k/k$ and $\Delta y/y$ are equal to zero for any technology level, but a higher technology level leads to higher k^* and y^*.

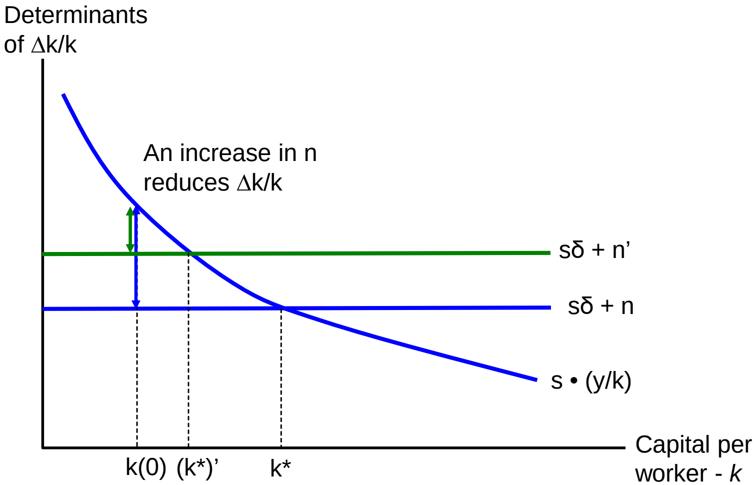
A Change in Labor



A Change in Labor

- What happens to $\Delta k/k$, k^* and y^* if there is a change in the amount of labor, L?
- In the short run, an increase in L raises $\Delta k/k$ and $\Delta y/y$, which remain higher during the transition to the steady state.
- In the long run, $\Delta k/k$ and $\Delta y/y$ are equal to zero for any L.
- Also, k^* and y^*, are the same for any L.

A Change in Population Growth



A Change in Population Growth

- What happens to $\Delta k/k$, k^* and y^* if there is a change in the population growth rate, n?
- In the short run, an increase in n reduces $\Delta k/k$ and $\Delta y/y$, which remain lower during the transition to the steady state.
- In the long run, $\Delta k/k$ and $\Delta y/y$ equal to zero for any, but a higher population growth rate leads to lower k^* and y^*.
- A change in the depreciation rate, δ, has the same effect.

Summary

Increase in	**Effect on k* and y***
Saving rate, s	INCREASE
Technology, A	INCREASE
Population growth, n	DECREASE
Depreciation	DECREASE
Labor input, L	NO EFFECT

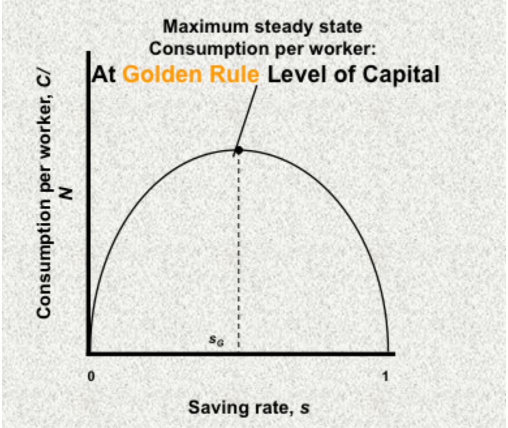
Optimum saving rate: Golden Rule

- Maximize consumption with respect to s subject to the steady state condition:

$$\max_{s} c^*$$

$$\begin{aligned} c^* &= y^* - i^* \\ &= y^* - sy^* \\ &= A(1-s) f(k^*) \\ &= (1-s) A \left(\frac{s}{g_n + \delta} \right)^{\frac{\alpha}{1-\alpha}} \end{aligned}$$

Optimum saving rate: Golden Rule



Optimum saving rate: Golden Rule

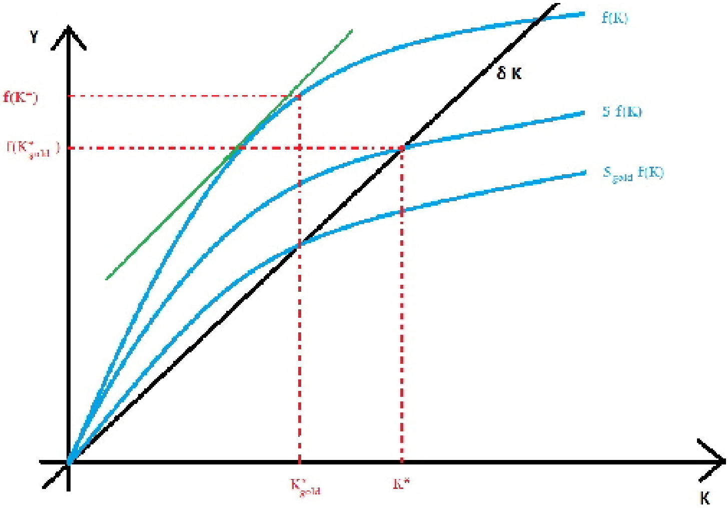
take logs:

$$\log c^* = \log(1-s) + \frac{\alpha}{1-\alpha}\log s + \log\frac{A}{(g_n+\delta)^{\frac{\alpha}{1-\alpha}}}$$

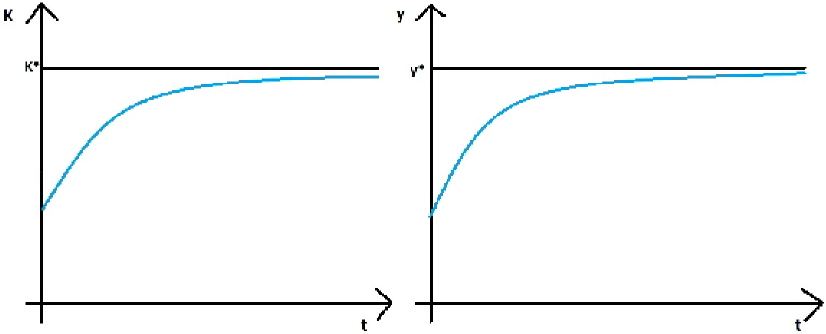
FOC:

$$\begin{aligned} \frac{-1}{1-s} + \frac{\frac{\alpha}{1-\alpha}}{s} &= 0 \\ \frac{\alpha}{1-\alpha}(1-s) &= s \\ s^* &= \alpha \end{aligned}$$

Optimum saving rate: Golden Rule



- The Biggest Failure of the model:
  - No sustained per capita growth of GDP and capital
  - Due to DECREASING RETURNS to scale.



Convergence

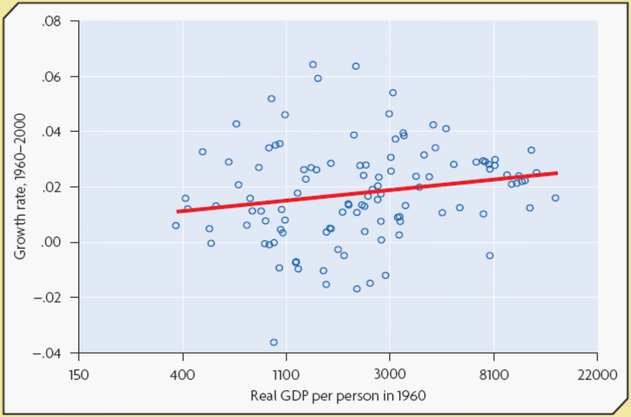
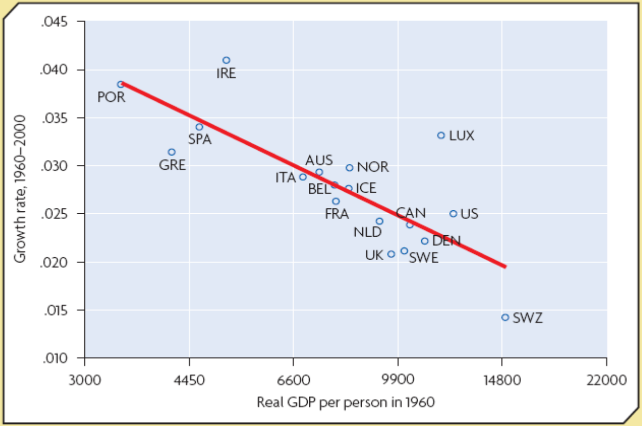


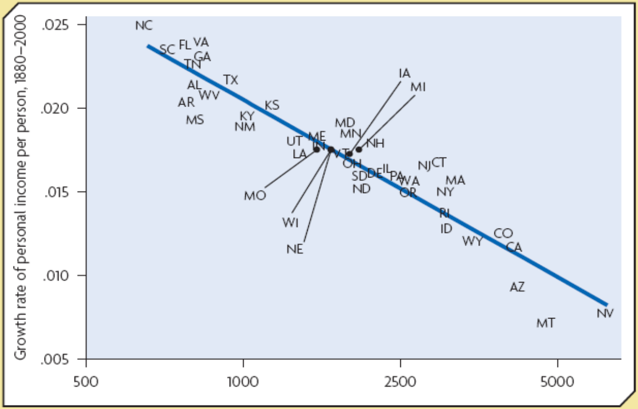
Figure 4.9 *Growth Rate Versus Level of Real GDP per Person for a Broad Group of Countries*

Figure 4.10 *Growth Rate Versus Level of Real GDP per Person for OECD Countries*

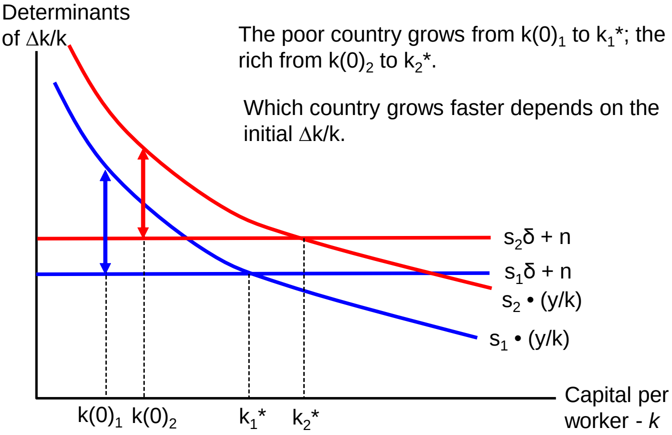


Convergence

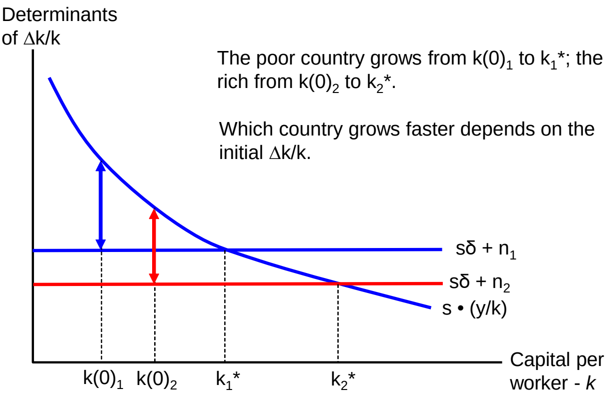
Figure 4.11 *Growth Rate Versus Level of Income per Person for U.S. States, 1880–2000*



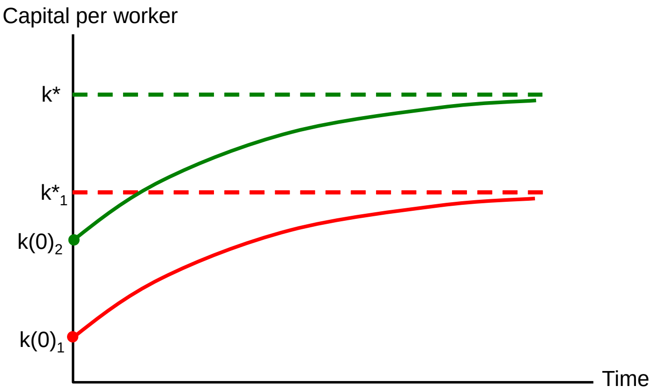
Conditional Convergence: Different Saving Rates



Conditional Convergence: Different Population Growth Rates



Transition Paths For Two Economies



Exogenous Growth

- Why don't we see sustained growth in the simple Solow model?
- How can we get it?

Exogenous Growth

- Exogenous productivity growth!!!
- How does it work?

$$\begin{aligned} \log Y_t &= \log A_t + \alpha \log K_t + (1-\alpha) \log L_t \\ g_Y &= g_A + \alpha g_K + (1-\alpha) g_L \end{aligned}$$

$\Rightarrow$

$$g_y = \underbrace{g_A}_{\text{Technical Growth}} + \underbrace{\alpha g_k}_{\text{Capital Deepening}}$$

- Intuition: How does productivity growth leads to capital deepening.

From $I = sY$ we had:

$$k_{t+1} = \frac{sy_t}{1+g_n} + \frac{1-\delta}{1+g_n} k_t$$

So

$$\frac{k_{t+1}}{k_t} = \left(\frac{s}{1+g_n}\right) \frac{y_t}{k_t} + \frac{1-\delta}{1+g_n}$$

Exogenous Growth

- Balanced Growth path: All variables have a constant growth rate.
- Thus $\frac{y_t}{k_t}$ is constant.
- Therefore y and k have the same growth rate; call it $g = g_Y = g_k \Rightarrow$

$$g = g_A + \alpha g$$

$\Rightarrow$

$$g = \frac{g_A}{1-\alpha}$$

- Amplification mechanism!
- Explain graphically!

Exogenous Growth

Also:

$$
\begin{aligned}
1+g &= \left(\frac{s}{1+g_n}\right)\frac{y_t}{k_t}+\frac{1-\delta}{1+g_n} \\
\frac{y_t}{k_t} &= \frac{(1+g)(1+g_n)-(1-\delta)}{s} \\
\frac{y_0}{k_0} &\approx \frac{g+g_n+\delta}{s} \\
A_0 k_0^{\alpha-1} &\approx \frac{g+g_n+\delta}{s} \\
k_0 &= \left(\frac{sA_0}{\frac{g_A}{1-\alpha}+g_n+\delta}\right)^{\frac{1}{1-\alpha}}
\end{aligned}
$$

Growth Accounting

- y_t and k_t are observable. How about A_t? How about α?
- Recall that with Cobb Douglas production function the shares of factor payments are constant:

$$\begin{aligned} Y_t &= rK_t + wL_t \\ \frac{wL_t}{Y_t} &= \alpha \end{aligned}$$

- Empirically we can test this:
  - In US: $\alpha \approx \frac{1}{3}$
  - In Iran: $\alpha \approx \frac{2}{3}$

Growth Accounting

- Now we know $a \Rightarrow$

$$\log A_t = \log y_t - \alpha \log k_t$$

- A_t computed this way is called the SOLOW RESIDUAL.
- It comes from Solow (1957) growth accounting framework.
  - He applied this framework to United States data.
  - Found that changes in A_t are responsible for $\approx$80% of changes in y_t .
  - Thus, changes in k_t are responsible for ONLY 20%!

Growth Accounting

- More sophisticated studies include more inputs than capital:
    - Male and female labor force participation.
    - Education.
    - Land and natural resources.
- For specific episodes the importance of input accumulation can be even higher:
    - The USSR had input accumulation as the main growth strategy.
    - Young (95), Krugman (94) and others provide evidence that the large post-war growth in Honk Kong, Singapore, South Korea and Taiwan was driven by input accumulation.
    - Hsieh (99) contest their ndings

Growth Accounting

Country	$\log \frac{y_{t+1}}{y_t}$	$\alpha \log \frac{k_{t+1}}{k_t}$	$\log \frac{A_{t+1}}{A_t}$
US (1929-66)	2.6	0.5	2.1
USSR (1928-700	2.7	1.5	1.2
Singapore	4.7	3.34	1.6

- Note: This does not include other inputs.
- Source: Ofer (1987), Bosworth and Collins (2003)

Growth Accounting

- What is A_t?
- Main problem with this methodology:
  - Moses Abramovitz (1956): At is the measure of our ignorance .
- Recall
  - In our theory A_t is EXOGENOUS.
  - In the data it is the part of $\frac{y_{t+1}}{y_t}$ not explained by k_t.
  - Yet, it is fundamental to understand growth.
- Opennig the black box of $A_t \Rightarrow$ Endogenous Growth models (new growth theory)

Issues in the Solow model

- Why is not the Solow model enough to understand growth?
- Why should the investment rate s be constant?
- How do people save or invest? How about firms?
- What happens in a decentralized economy? What heppens to the incentives?
- How do prices and interest rates affect the investment and production?