The Solow Growth Model

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March 1, 2021
1. In the data, real GDP/capita has been growing over time for most countries.

2. There are large differences in per capita income levels across countries: $34k vs $200.

3. In similar countries, economies with lower real GDP per capita had faster growth rates. But this is not true globally.

4. Some economies were converging to each other.

5. Still there are some stagnations.
Questions

1. What are the factors that lead to economic growth?
2. Why do some countries grow fast and others slow? (East Asians vs sub-Saharan African countries)
3. What policies increase real GDP per capita?
4. How did rich countries sustain growth rates of 2%?
Production function:

\[ Y_t = A_t \cdot F(K_t, L_t) \]

Capital accumulates through investment:

\[ K_{t+1} = (1 - \delta) K_t + I_t \]

and

\[ Y = C + I \]

\( I \) is the investment: Purchases of capital: In the data 14% of GDP.

\( F \) is diminishing marginal product of capital and labor.
F is constant returns to scale, therefore Real GDP per capita is

\[
\frac{Y}{L} = AF \left( \frac{K}{L}, \frac{L}{L} \right) \\
y = AF(k, 1) \\
y = Af(k)
\]
Example:

\[ Y = AK^\alpha L^{1-\alpha} \]

Therefore

\[ y = Ak^\alpha \]

As K increases, the marginal product of K (slope) decreases.
\[ \log Y_t = \log A_t + \alpha \log K_t + (1 - \alpha) \log L_t \]

\[ g_Y = g_A + \alpha g_K + (1 - \alpha) g_L \]

\[ \frac{\Delta Y}{Y} = \frac{\Delta A}{A} + \alpha \frac{\Delta K}{K} + (1 - \alpha) \frac{\Delta L}{L} \]

or

\[ \frac{\Delta y}{y} = \frac{\Delta A}{A} + \alpha \frac{\Delta k}{k} \]
Simplifying assumption

\[ I_t = sY_t \]

national savings rate: \( s \)
Population growth rate: $g_n$

Rewriting the law of motion:

\[
\begin{align*}
K_{t+1} &= (1 - \delta) K_t + sY_t \\
\frac{K_{t+1}}{L_{t+1}} \cdot \frac{L_{t+1}}{L_t} &= (1 - \delta) \frac{K_t}{L_t} + s \frac{Y_t}{L_t} \\
k_{t+1} (1 + g_n) &= (1 - \delta) k_t + sy_t \\
k_{t+1} &= \frac{sy_t}{1 + g_n} + \frac{1 - \delta}{1 + g_n} k_t \\
k_{t+1} &= \frac{sAf(k_t)}{1 + g_n} + \frac{1 - \delta}{1 + g_n} k_t = G(k_t)
\end{align*}
\]
Capital choices and capital dynamics

\[ G(K_t) \]

\[ K_{t+1} \]

\[ K_0, K_1, K_2, K_\infty \]
Average product of capital

Output per worker - $y$

Capital per worker - $k$

$A f(k)$

Average product of capital with $k_a$

Average product of capital with $k_b$

$A f(k)$
Capital growth rate

\[ \Delta k/k > 0 \text{ since } s \left( \frac{Y}{K} \right) > s\delta + n \]
\[ \frac{\Delta Y}{Y} = \frac{\Delta A}{A} + \alpha \frac{\Delta K}{K} + (1 - \alpha) \frac{\Delta L}{L} \]
\[ \frac{\Delta k}{k} = s \left( \frac{y}{k} \right) - \delta - n \]
\[ \frac{\Delta y}{y} = \alpha \left( s \left( \frac{y}{k} \right) - \delta - n \right) \]
Predictions

- Convergence
- Data: There is conditional convergence: economies with similar characteristics converge
- Lower $K$, higher growth

![Graph showing capital per worker over time with points $k(0)_1$, $k(0)_2$, and $k^*$]
Steady state level

\[
k^* = G(k^*) = \frac{sAf(k^*)}{1 + g_n} + \frac{1 - \delta}{1 + g_n} k^* \\
= \frac{sAk^*}{1 + g_n} + \frac{1 - \delta}{1 + g_n} k^* \\
= A^{\frac{1}{1-\alpha}} \left( \frac{s}{g_n + \delta} \right)^{\frac{1}{1-\alpha}} \\
i^* = \left( \frac{g_n + \delta}{1 + g_n} \right) k^* \\
y^* = A^{\frac{1}{1-\alpha}} \left( \frac{s}{g_n + \delta} \right)^{\frac{\alpha}{1-\alpha}} \\
Y_t = A^{\frac{1}{1-\alpha}} \left( \frac{s}{g_n + \delta} \right)^{\frac{\alpha}{1-\alpha}} L_0 (1 + g_n)^t 
\]
Comparative statics

- Changes in $s$, $g_n$, $A$, $\delta$, $\alpha$
A Change in the Saving Rate

For any level of $k$, $\Delta k/k$ is larger the higher the saving rate.

$\Delta k/k = s_1 \delta + n - k(0) k_1^*$

$\Delta k/k = s_2 \delta + n - s_2 \cdot \frac{y}{k} - s_1 \cdot \frac{y}{k}$

Determinants of $\Delta k/k$

$\text{Capital per worker} - k$

$k(0)$ $k_1^*$ $k_2^*$

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The Solow Growth Model

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A Change in the Saving Rate

- Compare two countries with different savings rates. How does that affect $\Delta k/k$, $k^*$ and $y^*$.
- A higher saving rate raises $\Delta k/k$, which rate remains higher during the transition to the steady state.
- In the long run, $\Delta k/k$ and $\Delta y/y$ are equal to zero for any saving rate, but a higher saving rate leads to higher steady-state $k^*$ and $y^*$. 
A Change in the Technology Level

The growth rate of capital per worker, \( \Delta k/k \), is higher at any \( k \) when the technology level is higher.

Determinants of \( \Delta k/k \)

- Capital per worker - \( k \)
- \( sA_1 \cdot f(k)/k \)
- \( sA_2 \cdot f(k)/k \)
- \( s\delta + n \)
What happens to $\Delta k/k$, $k^*$ and $y^*$ if there is a change in technology?

In the short run, an increase in $A$ raises $\Delta k/k$ and $\Delta y/y$, which remain higher during the transition to the steady state.

In the long run, $\Delta k/k$ and $\Delta y/y$ are equal to zero for any technology level, but a higher technology level leads to higher $k^*$ and $y^*$. 
A Change in Labor

Determinants of $\Delta k/k$

An increase in $L_0$ reduces $k_0$ and increases $\Delta k/k$

$\frac{s\delta + n}{s \cdot \frac{y}{k}}$

$\frac{k(0)'}{k(0)}$  $k(0)$  $k^*$

Capital per worker - $k$
What happens to $\Delta k/k$, $k^*$ and $y^*$ if there is a change in the amount of labor, $L$?

In the short run, an increase in $L$ raises $\Delta k/k$ and $\Delta y/y$, which remain higher during the transition to the steady state.

In the long run, $\Delta k/k$ and $\Delta y/y$ are equal to zero for any $L$.

Also, $k^*$ and $y^*$, are the same for any $L$. 
A Change in Population Growth

Determinants of $\Delta k/k$

An increase in $n$ reduces $\Delta k/k$

$\Delta k/k = s\delta + n$ (blue line)

$\Delta k/k = s\delta + n'$ (green line)

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The Solow Growth Model

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What happens to $\Delta k/k$, $k^*$ and $y^*$ if there is a change in the population growth rate, $n$?

In the short run, an increase in $n$ reduces $\Delta k/k$ and $\Delta y/y$, which remain lower during the transition to the steady state.

In the long run, $\Delta k/k$ and $\Delta y/y$ equal to zero for any, but a higher population growth rate leads to lower $k^*$ and $y^*$.

A change in the depreciation rate, $\delta$, has the same effect.
<table>
<thead>
<tr>
<th>Increase in</th>
<th>Effect on $k^<em>$ and $y^</em>$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Saving rate, $s$</td>
<td>INCREASE</td>
</tr>
<tr>
<td>Technology, $A$</td>
<td>INCREASE</td>
</tr>
<tr>
<td>Population growth, $n$</td>
<td>DECREASE</td>
</tr>
<tr>
<td>Depreciation</td>
<td>DECREASE</td>
</tr>
<tr>
<td>Labor input, $L$</td>
<td>NO EFFECT</td>
</tr>
</tbody>
</table>
Maximize consumption with respect to $s$ subject to the steady state condition:

$$\max_s c^*$$

$$c^* = y^* - i^*$$
$$= y^* - sy^*$$
$$= A(1 - s)f(k^*)$$
$$= (1 - s)A \left( \frac{s}{g_n + \delta} \right)^{\frac{\alpha}{1-\alpha}}$$
Optimum saving rate: Golden Rule
Optimum saving rate: Golden Rule

take logs:

\[
\log c^* = \log (1 - s) + \frac{\alpha}{1 - \alpha} \log s + \log \frac{A}{(g_n + \delta)^{1-\alpha}}
\]

FOC:

\[
\frac{-1}{1 - s} + \frac{\alpha}{s} = 0
\]

\[
\frac{\alpha}{1 - \alpha} (1 - s) = s
\]

\[
s^* = \alpha
\]
Optimum saving rate: Golden Rule
Solow Model

The Biggest Failure of the model:
- No sustained per capita growth of GDP and capital
- Due to DECREASING RETURNS to scale.

\[ K(t) \]
\[ y(t) \]
Figure 4.9 Growth Rate Versus Level of Real GDP per Person for a Broad Group of Countries
Figure 4.10  Growth Rate Versus Level of Real GDP per Person for OECD Countries
Figure 4.11  Growth Rate Versus Level of Income per Person for U.S. States, 1880–2000
Conditional Convergence: Different Saving Rates

Determinants of $\Delta k/k$

The poor country grows from $k(0)_1$ to $k_1^*$; the rich from $k(0)_2$ to $k_2^*$.

Which country grows faster depends on the initial $\Delta k/k$.

$s_1\delta + n$

$s_2\delta + n$

$s_1 \cdot (y/k)$

$s_2 \cdot (y/k)$

$k(0)_1$ $k(0)_2$ $k_1^*$ $k_2^*$

Capital per worker - $k$
Conditional Convergence: Different Population Growth Rates

The poor country grows from $k(0)_1$ to $k_1^*$; the rich from $k(0)_2$ to $k_2^*$.

Which country grows faster depends on the initial $\Delta k/k$. 

Determinants of $\Delta k/k$

Capital per worker $- k$

$\frac{s\delta + n_1}{k}$

$\frac{s\delta + n_2}{k}$

$s \cdot \frac{y}{k}$

$k(0)_1$ $k(0)_2$ $k_1^*$ $k_2^*$
Transition Paths For Two Economies

Capital per worker vs. Time

- Capital per worker: $k^*$, $k^*_1$, $k(0)_2$, $k(0)_1$
- Time: $t$

Diagram showing the capital per worker over time for two economies, with initial capital per worker levels $k(0)_1$ and $k(0)_2$, and steady-state capital per worker levels $k^*$ and $k^*_1$. The graph illustrates the transition paths for each economy, indicating how capital per worker evolves over time.
Why don’t we see sustained growth in the simple Solow model?

How can we get it?
Exogenous Growth

- Exogenous productivity growth!!!
- How does it work?

\[
\begin{align*}
\log Y_t &= \log A_t + \alpha \log K_t + (1 - \alpha) \log L_t \\
g_Y &= g_A + \alpha g_K + (1 - \alpha) g_L
\end{align*}
\]

\[
\Rightarrow 
\[
\begin{align*}
g_y &= g_A + \alpha g_K \\
&= \underbrace{g_A}_{\text{Technical Growth}} + \underbrace{\alpha g_K}_{\text{Capital Deepening}}
\end{align*}
\]

- Intuition: How does productivity growth leads to capital deepening.
Exogenous Growth

From $I = sY$ we had:

$$k_{t+1} = \frac{sy_t}{1+g_n} + \frac{1-\delta}{1+g_n}k_t$$

So

$$\frac{k_{t+1}}{k_t} = \left(\frac{s}{1+g_n}\right)\frac{y_t}{k_t} + \frac{1-\delta}{1+g_n}$$
Balanced Growth path: All variables have a constant growth rate.
Thus $\frac{y_t}{k_t}$ is constant.
Therefore $y$ and $k$ have the same growth rate; call it $g = g_Y = g_k \Rightarrow$

$$g = g_A + \alpha g$$

$$\Rightarrow \quad g = \frac{g_A}{1 - \alpha}$$

Amplification mechanism!
Explain graphically!
Exogenous Growth

Also:

\[
1 + g = \left( \frac{s}{1 + g_n} \right) \frac{y_t}{k_t} + \frac{1 - \delta}{1 + g_n} \\
y_t \frac{y_t}{k_t} = \frac{(1 + g)(1 + g_n) - (1 - \delta)}{s} \\
y_0 \frac{y_0}{k_0} \approx \frac{g + g_n + \delta}{s} \\
A_0 k_0^{\alpha - 1} \approx \frac{g + g_n + \delta}{s} \\
k_0 = \left( \frac{sA_0}{\frac{gA}{1-\alpha} + g_n + \delta} \right)^{\frac{1}{1-\alpha}}
\]
\( y_t \) and \( k_t \) are observable. How about \( A_t \)? How about \( \alpha \)?

Recall that with Cobb Douglas production function the shares of factor payments are constant:

\[
Y_t = rK_t + wL_t
\]

\[
\frac{wL_t}{Y_t} = \alpha
\]

Empirically we can test this:

- In US: \( \alpha \approx \frac{1}{3} \)
- In Iran: \( \alpha \approx \frac{2}{3} \)
Now we know $a \Rightarrow$

$$\log A_t = \log y_t - \alpha \log k_t$$

$A_t$ computed this way is called the SOLOW RESIDUAL.

It comes from Solow (1957) growth accounting framework.

- He applied this framework to United States data.
- Found that changes in $A_t$ are responsible for $\approx 80\%$ of changes in $y_t$.
- Thus, changes in $k_t$ are responsible for ONLY 20\%! 
More sophisticated studies include more inputs than capital:
- Male and female labor force participation.
- Education.
- Land and natural resources.

For specific episodes the importance of input accumulation can be even higher:
- The USSR had input accumulation as the main growth strategy.
- Young (95), Krugman (94) and others provide evidence that the large post-war growth in Honk Kong, Singapore, South Korea and Taiwan was driven by input accumulation.
- Hsieh (99) contest their findings.
Growth Accounting

<table>
<thead>
<tr>
<th>Country</th>
<th>$\log \frac{y_{t+1}}{y_t}$</th>
<th>$\alpha \log \frac{k_{t+1}}{k_t}$</th>
<th>$\log \frac{A_{t+1}}{A_t}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>US (1929-66)</td>
<td>2.6</td>
<td>0.5</td>
<td>2.1</td>
</tr>
<tr>
<td>USSR (1928-700)</td>
<td>2.7</td>
<td>1.5</td>
<td>1.2</td>
</tr>
<tr>
<td>Singapore</td>
<td>4.7</td>
<td>3.34</td>
<td>1.6</td>
</tr>
</tbody>
</table>

- **Note:** This does not include other inputs.
- **Source:** Ofer (1987), Bosworth and Collins (2003)
Growth Accounting

- What is $A_t$?
- Main problem with this methodology:
  - Moses Abramovitz (1956): $A_t$ is the measure of our ignorance.
- Recall
  - In our theory $A_t$ is EXOGENOUS.
  - In the data it is the part of $\frac{y_{t+1}}{y_t}$ not explained by $k_t$.
  - Yet, it is fundamental to understand growth.
- Opening the black box of $A_t \Rightarrow$ Endogenous Growth models (new growth theory)
Issues in the Solow model

- Why is not the Solow model enough to understand growth?
- Why should the investment rate \( s \) be constant?
- How do people save or invest? How about firms?
- What happens in a decentralized economy? What happens to the incentives?
- How do prices and interest rates affect the investment and production?