# Static Games with Incomplete Information

GholamReza Keshavarz Haddad

Sharif University of Technology
Graduate School of Management and Economics

April 13, 2020

# Outline

# Introduction

- So far, we have studied games with Nash ( Pure or Mixed) Equilibria and games with Bayesian perfect Equilibria

Timing

| Information | | Simultaneous | Sequential |
|---|---|---|---|
| | Complete | **Nash E., pure or mixed** | ? |
| | Incomplete | **Bayesian Nash E., p. or m.** | ? |

# Introduction

- So far, we have focused on games in which any piece of information that is known by any player is known by all the players (and indeed common knowledge).

- Such games are called the games of complete information.
    - In the games with mixed strategies, any of players does not have informational advantageous, common knowledge.

- In real life, players always have some private information that is not known by other parties.

# Introduction

### Example (Payoff with type parameter)

We can hardly know other players' preferences. Imagine a situation with two players whose Bernoulli utility functions are $u_1(s_1, s_2, \theta_1)$ and $u_2(s_1, s_2, \theta_2)$. Where the $\theta_1$ and $\theta_2$, type of their preferences and are private information.

- In these cases a party may have some information that is not known by some other party.
- Such games are called games of incomplete information or asymmetric information.

### Example

- Recall the Cournot duopoly equilibrium, with $b = 1$.
- Aggregate inverse demand is given by $p = a - (q_1 + q_2)$, and the total production cost for the firm 1 is $cq_1$.
- Firm 2 can use two technology in production line: $c_H q_2$, and $c_L q_2$ with probability of $\mu$ and $(1 - \mu)$, respectively, where $c_L < c_H$.

### Example

- Information is asymmetric: Firm 2 knows its own technology and that of firm 1's, but firm 2 its own production technology and only that firm 2 may use technology $H$ with probability $\mu$ and technology $L$ with probability $1 - \mu$.

- Thus, the probability distribution of the production technologies and $c_L < c_H$ are common knowledge

# A Motivating Example: Cournot duopoly with asymmetric information, CONT.

- If Firm 2's cost function is high, it will choose $q_2^*(c_H)$ to solve firm 2 is:

$$max_{q_2} \quad [a - \bar{q}_1 - q_2 - c_H]q_2 \tag{1}$$

- If Firm 2's cost function is low, it will choose $q_2^*(c_L)$ to solve firm 2 is:

$$max_{q_2} \quad [a - \bar{q}_1 - q_2 - c_L]q_2 \tag{2}$$

- Give the common knowledge about the technology types of Firm 2, the Firm 1 chooses $q_1^*$) to solve:

$$max_{q_1} \mu.[a - q_1 - q_2^*(c_H) - c]q_1 \tag{3}$$

$$+(1-\mu).[a - q_1 - q_2^*(c_L) - c]q_1$$

- The F.O.C for these three objective functions are:

$$q_2^*(c_H) = \frac{a - q_1^* - c_H}{2} \quad (4)$$

$$q_2^*(c_L) = \frac{a - q_1^* - c_L}{2} \quad (5)$$

and

$$q_1^* = \frac{\mu[a - q_2^*(c_H) - c] + (1-\mu)[a - q_2^*(c_L) - c]}{2} \quad (6)$$

$$q_1^* = \frac{a - c - E[q_2^*]}{2}$$

- The solution for these F.O.Cs ( or reaction functions) are:

## A Motivating Example: Cournot duopoly with asymmetric information, CONT.

- The solution for these F.O.Cs ( or reaction functions) are:

$$q_2^*(c_H) = \frac{a - 2c_H + c}{3} + \frac{1-\mu}{6}(c_H - c_L) \tag{7}$$

$$q_2^*(c_L) = \frac{a - 2c_L + c}{3} - \frac{\mu}{6}(c_H - c_L) \tag{8}$$

and

$$q_1^* = \frac{a - 2c + \mu c_H + (1-\mu)c_L}{3} \tag{9}$$

$$q_1^* = \frac{a - 2c + E[c_2]}{3} \tag{10}$$

- Why the decision rule $q_2^*(c_H)$ is a function of $c_L$, or $q_2^*(c_L)$ is a function $c_H$?

# A Motivating Example: Cournot duopoly with asymmetric information, CONT.

- Player 2 does know that The Player 1 does not know by which technology Firm 2 is going to produce.
- While Firm 2 deciding about its type choice (H or L), it takes into account this uncertainty of Firm 1.
- How do you compare the solution with those of Nash-Cournot equilibrium $q_c = (a - c)/3$?
- Assume that we have only one type for Firm 2, namely, $c_2 = c_H = c_L$ and $c_1 = c$ for Firm 1.

# Extensive and strategic form of a games with incomplete information

## Example

**Prisoners' Dilemma with Incomplete Information**

- Consider the modified version of prisoners' dilemma in which, with probability $\mu$ prisoner 2 has preference (not rat) $\theta_1$ and probability of $1 - \mu$ for ratting $\theta_2$ on his accomplice.

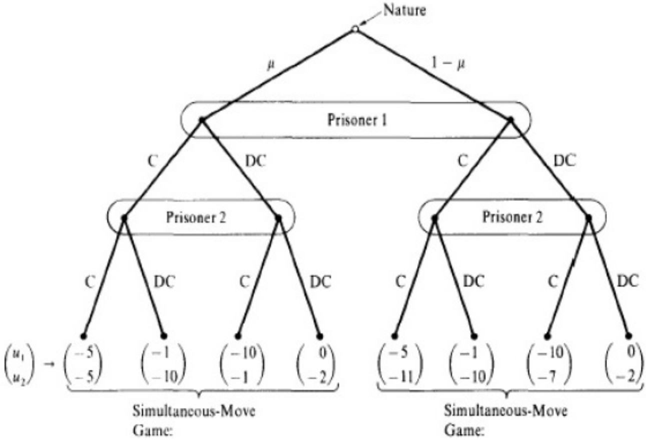
- Ratting will cause 6 units of dis-utility for P2, he is not a bad guy!

- Set of prisoner 2's types is $\Theta_2 = \{\theta_1, \theta_2\} = \{0, 6\}$, whose distribution is common knowledge.

# Extensive form of the game

**Prisoners' Dilemma with Incomplete Information** , cont.

- The extensive form game is represented for the players by *DC* and *C*, which stand for "*Don't Confess*" and "*Confess*", respectively



# Strategic form of the game

**Prisoners' Dilemma with Incomplete Information** , cont.

- Prisoner two has two strategies and two types, we can represent his strategy function as $s_2(\theta)$
- His complete contingent plan is:
  - $C(\theta_1)$, $C(\theta_2)$
  - $C(\theta_1)$, $DC(\theta_2)$
  - $DC(\theta_1)$, $C(\theta_2)$
  - $DC(\theta_1)$, $DC(\theta_2)$
- Recall that types set of **P2** is $\Theta_2 = \{\theta_1, \theta_2\} = \{0, 6\}$

| | | P2 | |
|---|---|---|---|
| | | DC | C |
| | DC | 0, -2 | -10, -1-$\theta_1$ |
| P1 | C | -1, -10 | -5, -5-$\theta_1$ |

| | | P2 | |
|---|---|---|---|
| | | DC | C |
| | DC | 0, -2 | -10, -1-$\theta_2$ |
| P1 | C | -1, -10 | -5, -5-$\theta_2$ |

# Strategic form of the game

**Prisoners' Dilemma with Incomplete Information** , cont.

- For pedagogical purpose and ease of presentation, I used two separated payoff matrices to show the incompleteness of information
- Game theory literature, by convention, one payoff matrix with unknown parameters is used
- since one of the players has two types of preference, applying one notation $\theta \in \{0, 6\}$ is enough

P2

| | | DC | C |
|---|---|---|---|
| P1 | DC | 0, -2 | -10, -1-$\theta$ |
| | C | -1, -10 | -5, -5-$\theta$ |

# Bayesian Nash Equilibrium

- Player $i$'s payoff function $u_i(s_i, s_{-i}, \theta_i)$, where $\theta_i \in \Theta_i$ is a random variable.

- The joint distribution of $\theta_i$'s is given by $F(\theta_1, ..., \theta_I)$, which is common knowledge among the players

- Given the notations, a Bayesian game is represented by:

$$[I, \{S_i\}, \{u_i(.)\}, \Theta, F(.)]$$

- Set of all possible types for all players is $\Theta = \Theta_1 \times, ..., \times \Theta_I$

# Bayesian Nash Equilibrium

- A **Bayesian Nash equilibrium** is simply a **Nash equilibrium** in a **Bayesian game**.

### Definition (Pure Strategy Bayesian Nash Equilibrium)

In the static Bayesian game $[I, \{S_i\}, \{u_i(.)\}, \Theta, F(.)]$ the strategies $s^* = (s_1^*, ..., s_I^*)$ are a pure strategy Bayesian Nash Equilibrium if for each player $i$ and for each of $i$'s types $\theta_i \in \Theta_i$,types the action $s^*(\theta_i)$ solves:

$$s_i^*(\theta_i) =$$

$$\underset{s_i \in S_i}{\operatorname{argmax}} \sum_{\theta_{-i} \in \Theta_{-i}} u_i[s_1^*(\theta_1), ..., , s_{i-1}^*(\theta_{i-1}), s_i, s_{i+1}^*(\theta_{i+1}), ...,$$

$$s_I^*(\theta_I)|\bar{\theta}_i]p(\theta_{-i}|\bar{\theta}_i)$$

- $p(\theta_{-i}|\bar{\theta}_i) = p(\theta_{-i})$ if the $(\theta_{-i}$ is independent of $\theta_i$, like the $Pr(\theta_1) = \mu$ in the prisoner's dilemma.

# Bayesian Nash Equilibrium

- Recall the optimal solution (7) to (10) [*I retyped for the ease of communication in below*], in which Firm 2's optimal strategy is depend on its type.
- The optimal strategy of firm 1 depends only on the Expected value of its rival's types, instead.
- Firm 2 will choose either $q_2^*(c_H) = \frac{a-2c_H+c}{3} + \frac{1-\mu}{6}(c_H - c_L)$ or $q_2^*(c_L) = \frac{a-2c_L+c}{3} - \frac{\mu}{6}(c_H - c_L)$, subject to its value function of profit.

$$q_1^* = \frac{a - 2c + \mu c_H + (1-\mu)c_L}{3}$$

$$q_1^* = \frac{a - 2c + E[c_2]}{3}$$

- Player 1 has only one type $c$, therefore she has only one $s_1^*(c)$ function of her own type

# Bayesian Nash Equilibrium

- For the continuous and i.i.d preference types $\Theta_{-i}$ with the joint density function of $f(\theta_{-i})$, the conditional expected utility function for player $i$ in concise form is:

$$s_i^*(\theta_i) = \underset{s_i \in S_i}{\operatorname{argmax}} \int \cdots \int_{\Theta_{-i}} u_i(s_i, s_{-i}^*(\theta_{-i}))|\bar{\theta}_i) f(\theta_{-i}) d\theta_{-i}$$

**Theorem**

*A profile of decision rules* $(s_1(.), ..., s_I(.))$ *( equations 7-9) is a Bayesian Nash equilibrium game*$[I, \{S_i\}, \{u_i(.)\}, \Theta, F(.)]$ *if only if , for all* $i$ *and for all* $\bar{\theta}_i \in \Theta_i$ *occurring with positive probability*

$$E_{\theta_{-i}}[u_i(s_i(\bar{\theta}_i), s_{-i}(\theta_{-i}), \bar{\theta}_i)|\bar{\theta}_i] \geq E_{\theta_{-i}}[u_i(s_i', s_{-i}(\theta_{-i}), \bar{\theta}_i)|\bar{\theta}_i]$$

*for all* $s_i' \in S_i$*, where the expectation is taken over realization of the other players' r.v. [the types, recall equation 3] conditional on player* $i$*'s realized signal* $\bar{\theta}_i$*.*

# Bayesian Nash Equilibrium

- Literately, the theorem says, player $i$ chooses the action that maximizes his expected payoff.

- The expected payoff uses conditional distribution of the all rivals' types.

- Conditional distribution of the types $\theta$ is

$$F(\theta_{-i}|\theta_i) = \frac{F(\theta_i, \theta{-}i)}{F(\theta_i)}$$

  Which is called in probability theory the **Bayes Rule**

- If the types are independently distributed, (recall the prisoners' dilemma), then the conditional probability distribution function reduces to unconditional, $F(\theta_{-i}|\theta_i) = F(\theta{-}i)$.

# Bayesian Nash Equilibrium, prisoners' dilemma

- Rationality requires the prisoner two to play the dominant strategy for each realized type.

- He plays $C$ if $\theta_1$ is realized by nature ( the third player) as his dominant strategy
- He plays $DC$ if $\theta_2$ is realized by nature as his dominant strategy

- Which strategy should prisoner one choose?
- He should compare the expected payoffs of $DC$ and $C$.

$$E[u_1(s_1, s_2(.))|s_1 = DC] = (\mu)(-10) + (1-\mu)(0)$$

$$E[u_1(s_1, s_2(.))|s_1 = C] = (\mu)(-5) + (1-\mu)(-1)$$

$$E[u_1(s_1, s_2(.))|s_1 = DC] \geq E[u_1(s_1, s_2(.))|s_1 = C]$$

- prisoner 1 prefers $DC$ over $C$ if he believes that $\mu \leq \frac{1}{6}$

## Example (Battle of the Sexes)

- Remember that in the Battle of the Sexes, a husband and a wife were deciding to go for watching *Ballet* or *Box*.
- They both would rather spend the evening together than apart
- Now suppose that although they have known each other for quite some time, Christina and Patrick aren't sure of each other's payoffs
- **A technical note:** $p(t < \bar{\theta}) = \int_0^{\bar{\theta}} (1/x) dx = \bar{\theta}/x$

| | | **Patrick** | |
|---|---|---|---|
| | | *Ballet* $\bar{\theta}_p/x$ | *Box* $(1\text{-}\bar{\theta}_p/x)$ |
| **Christina** | *Ballet* $(1\text{-}\bar{\theta}_c/x)$ | $2+t_c$, 1 | 0, 0 |
| | *Box* $\bar{\theta}_c/x$ | 0, 0 | 1, $2+t_p$ |

## Example (Battle of the Sexes, Cont.)

- Suppose that Christina's payoff if both attend the opera is $2+t_c$, where $t_c$ is privately known by Christina, and Patrick's payoff if both attend the Box is $2+t_p$, where $t_p$ is privately known by Patrick
- $t_c$ and $t_p$ are independent draws from a uniform distribution on $[0, x]$.
- The action spaces are $\mathcal{A}_c = \mathcal{A}_p = \{Ballet, Box\}$
- The type spaces are $\Theta_c = \Theta_p = [0, x]$

## Example (Battle of the Sexes, Cont.)

- Christina plays *Ballet* if $t_c$ exceeds a critical value $\bar{\theta}_c$ and plays *Box* otherwise.
- Patrick plays *Box* if $t_p$ exceeds a critical value $\bar{\theta}_p$ and plays *Ballet* otherwise.
- Given Patrick's strategy, Christina's expected payoffs from playing *Ballet* and *Box* respectively are:

$$u_c(Ballet, s_p(\theta_p)) = (\bar{\theta}_p/x)(2 + t_c) + 0 \times (1 - \bar{\theta}_p/x)$$

$$u_c(Box, s_p(\theta_p)) = (\bar{\theta}_p/x) \times 0 + 1 \times (1 - \bar{\theta}_p/x)$$

- Which action should Christina take to maximize her expected utility function?

## Example (Battle of the Sexes, Cont.)

- Playing *Ballet* is only optimal if,

$$t_c \geq (x/\bar{\theta}_p) - 3 = \bar{\theta}_c$$

- In a similar manner one can find Patrick's expected payoffs' from playing *Box* and *Ballet*, finally:

$$t_p \geq (x/\bar{\theta}_c) - 3 = \bar{\theta}_p$$

- Solving these two optimal strategies simultaneously leads to $\bar{\theta}_p = \bar{\theta}_c$ and $\bar{\theta}_p^2 + 3\bar{\theta}_p - x = 0$, $\quad \bar{\theta}_p = \frac{-3 \pm \sqrt{9+4x}}{2}$
- Remember that $\theta_i$ is non-negative, ignore the negative root.

## Example (Battle of the Sexes, Cont.)

- The probability that Christina plays *Ballet*, namely $(1 - \bar{\theta}_c/x)$.
- The probability that Patrick plays *Box*, namely $(1 - \bar{\theta}_p/x)$.
- Solving that quadratic and substituting the solution in probabilities gives us that

$$Pr(t_c > \bar{\theta}_c) = 1 - \frac{-3 + \sqrt{9 + 4x}}{2x}$$

- Which approaches 2/3 as $x$ approaches zero, the mixed equilibrium!
- The players' behavior in this pure strategy Bayesian Nash equilibrium of the incomplete-information game approaches to the mixed-strategy Nash equilibrium in the original game of complete information.

## Example (Zigger Project)

- Two firms jointly share their research outputs. Each firm can independently choose to spend $c \in (0, 1)$ to develop the *zigger*, a device that is then made available to the other firm.

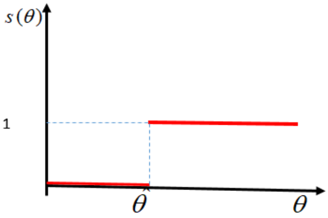
- Firm $i$'s type is $\theta_i$, which is believed by firm $-i$ to be independently drawn from the uniform distribution on $[0, 1]$.

- The benefit of the *zigger* when the type is $\theta_i$ is $\theta_i^2$.

- The timing is: the two firms privately observe their own type. Then they each simultaneously choose either to develop the *zigger* or not.

## Example (Zigger Project, Cont.)

- Value of the *zigger* to firm $i$ if it use the Zigger but not provided: $\theta_i^2$
- Payoff if the *zigger* is not provided: 0
- Payoff if it builds the *zigger* and apply it: $\theta_i^2 - c$
- payoff if it does not build the *zigger* but firm $-i$ does: $\theta_i^2$
- $s_i : [0, 1] \to \{yes(1), no(0)\}$



## EXAMPLE (ZIGGER PROJECT, CONT.)

- Let $p_{-i} = p(s_{-i}(\theta_{-i}) = 1)$ or $[p_2 = p(s_2(\theta_2) = 1)]$ denotes the probability that firm $-i$ produces the *zigger*, given its type $\theta_{-i}$.
- **Solve for the Pure Strategy Nash Equilibrium**
- Payoff matrix for game is:

| | | $-i$: *0* $\quad 1 - p_{-i}(s_{-i} = 1)$ | $-i$: *1* $\quad p_{-i}(s_{-i} = 1)$ |
|---|---|---|---|
| $i$ | *0* $\quad 1 - p_i(s_i = 1)$ | $0, 0$ | $\theta_i^2, \theta_{-i}^2 - c$ |
| | *1* $\quad p_i(s_i = 1)$ | $\theta_i^2 - c, \theta_{-i}^2$ | $\theta_i^2 - c, \theta_{-i}^2 - c$ |

## Example (Zigger Project, cont.)

- $\theta_i$s are *i.i.d* $\forall i \in \{1, 2\}$, with uniform distribution $[0, 1]$
- Firm $i$ should provide the *zigger* only if payoff from provision $\theta_i^2 - c$ is more than $p_{-i}(s_{-i} = 1)\theta_i^2$

$$\theta_i^2 - c \geq p_{-i}(s_{-i} = 1)\theta_i^2$$

- Equivalently, $\theta_i \geq \sqrt{\frac{c}{1 - p_{-i}(s_{-i}=1)}}$
- Suppose that firm $i$ and $-i$ use a cutoff strategy, $\hat{\theta}_i$ and $\hat{\theta}_{-i}$
- Technical note: $\int_0^{\hat{\theta}_i} d\theta_i = \hat{\theta}_i$ which is the probability of not developing the *Zigger* by $i$

## Example (Zigger Project, Cont.)

- Then, firm $i$ will provide the *zigger* with probability $1-\hat{\theta}_i = 1-\sqrt{\frac{c}{1-p_{-i}(s_{-i}=1)}} = 1-\sqrt{\frac{c}{\hat{\theta}_{-i}}}$
- Therefore $\hat{\theta}_i = \sqrt{c/\hat{\theta}_{-i}}$
- That is, $\hat{\theta}_i^2.\hat{\theta}_{-i} = c$
- and symmetrically, $\hat{\theta}_{-i}^2.\hat{\theta}_i = c$
- Canceling, $\hat{\theta}_i = \hat{\theta}_{-i}$, Thus, the only BNE is symmetric.
- Substituting into the equation above:$\hat{\theta}_i = \hat{\theta}_{-i} = c^{1/3}$

## Example (Zigger Project, cont.)

- **When firm $i$ can make free riding?**
- The zigger should be provided by one of the two firms if $\theta_i^2 \geq c$, then $\theta_i \leq c^{1/2}$.
- Given that $c \in (0, 1)$, we have that $c^{1/2} < c^{1/3}$.

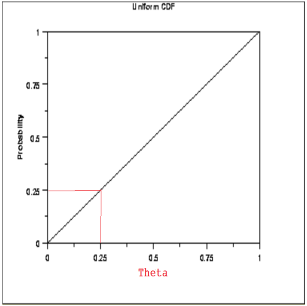



Figure: Uniform distribution function with $\theta \in [0, 1]$

## Example (War of attrition)

- A war of attrition is a situation where two players compete to see which is the first to quit the game.
- The player who stays longest wins the prize
- Wars of attrition occur in animal behavior (fighting over a territory), human behavior (see who stays the longest), interaction among firms (wait for another firm to exit an industry..)
- Formally, a war of attrition is like a second price auction where both the winner and the loser pay (this is called an *all-pay auction*)

## Example (War of attrition, Cont.)

- Suppose that players have a benefit from surviving the war of attrition, $\theta_i$ which is privately known.
- The value $\theta_i$ is distributed independently according to some distribution law, for example $p(.)$
- Each player $i, j$ chooses a time $s_i$ as a function of $\theta_i$ to exit.
- players decide about the value of $s_i$ and $s_j$ at the beginning of the game, but keep it as a private information
- Payoffs are:

$$u_i(s_i, s_j, \theta_i) = \begin{cases} -s_i & \text{if } s_i \leq s_j \\ \theta_i - s_j & \text{if } s_i > s_j \end{cases} \tag{11}$$

## Example (War of attrition, Cont.)

- What is the equilibrium strategy for player $i$? Basically, it comes form maximization of player's expected payoff respect to the strategy $s_i$, given her type.
- Expected payoffs for player $i$ is:

$$E[u_i(s_i, \theta_j|\theta_i)] = -s_i.Pr[s_i \leq s_j(\theta_j)] + \int_{\theta_j|s_i > s_j(\theta_j)} (\theta_i - s_j(\theta_j))f(\theta_j|\theta_i)d\theta_j \tag{12}$$

- We are looking for the $s_i^*(\theta_i)$ of this game which maximizes the conditional expected utility of player $i$.

## EXAMPLE (WAR OF ATTRITION, CONT.)

- The (pure-strategy) Bayesian equilibrium $(s_i(.), s_j(.))$ of this game. For each $\theta_i$, our derived strategy must satisfy $s_i(\theta_i)$ the following optimization problem:

$$s_i^*(\theta_i) \in \underset{s_i}{\operatorname{argmax}}\{-s_i.Pr[s_i \leq s_j(\theta_j)]$$

$$+\int_{\theta_j|s_i>s_j(\theta_j)} (\theta_i - s_j(\theta_j))f(\theta_j|\theta_i)d\theta_j\}$$

- Let's assume that $s_i(.)$ is an increasing and continuous function of $\theta_i$
- Then, the inverse function of $s_i = s_i(\theta_i)$ is re-presentable by $\theta_i = \Phi_i(s_i)$, and $s_i \leq s_j(\theta_j)$ is transformed to $\Phi_j(s_i) \leq \theta_j$.

## EXAMPLE (WAR OF ATTRITION, CONT.)

$$s_i^*(\theta_i) \in \underset{s_i}{\operatorname{argmax}}\{-s_i.[1 - P_j(\Phi_j(s_i))] + \int_0^{s_i} (\theta_i - s_j) f_j(\Phi_j(s_j))\Phi_j'(s_j) ds_j\} \tag{13}$$

- **Technical remarks**
  - If $f(x)$ and $x = g(z)$, then $f(z) = f(g^{-1}(x)).|dz/dx|$. So this clarifies why the $\Phi_j'(s_j)$ appears in (13).
  - $\theta_i$ in independent of $\theta_j$, therefore $f(\theta_j|\theta_i) = f(\theta_j)$
  - $\frac{d}{dx}\int_0^x f(t)dt = f(x)$
  - Derivative of first element of the objective function is: $\frac{d\{-s_i.[1-P_j(\Phi_j(s_i))]\}}{ds_i} = -[1 - P_j(\Phi_j(s_i))] + s_i f_j(\Phi_j(s_i))\Phi_j'(s_i)$
  - Derivative of second element of the objective function is: $(\Phi_i(s_i) - s_i) f(\Phi_j(s_i))\Phi_i'(s_i)$, where $\theta_i = \Phi_i(s_i)$

### Example (War of attrition, Cont.)

- F.O.C for the above maximization programming respect to the (upper limit of integral) decision variable $s_i$ is:

$$[1 - P_j(\Phi_j(s_i))] - \Phi_i(s_i) f_j(\Phi_i(s_i))\Phi_i'(s_i) = 0 \qquad (14)$$

- First term shows the marginal cost of an incremental change in $s_i$ and the second one is its marginal benefit.

## Example (War of attrition, Cont.)

- Suppose that $P_1 = P_2 = P$ and we are looking for a symmetric equilibrium.
- Substituting $\theta = \Phi(s)$ in equation (14), and using the fact that $\Phi' = 1/s'$, we have

$$s'(\theta) = \frac{\theta f(\theta)}{1 - P(\theta)}$$

or

$$s(\theta) = \int_0^{\theta} \left( \frac{x f(x)}{1 - P(x)} \right) dx$$

- Type with 0 value for the good are unwilling to fight for it, thus the lower limit of the integral equals zero.
- The optimal Bayesian Nash strategy is a function of $\theta$, as the PBNE definition implies.

### Example (War of attrition, Cont.)

- As an example, one can take the $P(\theta) = 1 - exp(-\theta)$, then the optimal strategy would be $s(\theta) = \frac{\theta^2}{2}$, which is a function of player's type $\theta$.
- Examine the ranges of type for $\theta < 2$ and $\theta > 2$. It is clear that for the latter $s(\theta) > \theta$.
  See *Fudenberg and Tirol*, page 219.

## Example (Public goods provision)

- Consider the following game of public good provision with private costs $c_i \geq 0$, with following payoff matrix:

| | | *Player 2* | |
|---|---|---|---|
| | | *Contribute* | *Don't Contribute* |
| *Player 2* | *Contribute* | 1-$c_1$, 1-$c_2$ | 1-$c_1$, 1 |
| | *Don't Contribute* | 1, 1-$c_2$ | 0, 0 |

- The cost $c_i$ is *i.i.d.* distributed with a uniform density on $\Theta_i = [0, 2]$, or $F(c_i) = \int_0^{c_i} \frac{1}{2-0} d\theta_i$.

## Example (Public goods provision cont.)

- Let type $c_i$ of player $i$ contributing be denoted by $s_i(c_i) = 1$, and not contributing by $s_i(c_i) = 0$.
- Then net utility is:

$$u_i(s_1(c_1), s_2(c_2), c_1, c_2) = max\{s_1(c_1), s_2(c_2)\} - c_i.s_i(c_i)$$

- Mixed strategy $\sigma_i$ for player $i$ in this game is given by $\sigma_i : \Theta_i \to \Delta(S_i)$
- Where $\Theta_i = [0, 2]$ and $S_i = \{0, 1\}$

## Example (Public goods provision cont.)

1. Compute a Bayesian Nash equilibrium of this game in pure strategies.
   - A strategy profile $s_i^*$ is a (pure strategy) BNE if $s_i^*(c_i)$ maximizes

     $$s_i^*(c_i) = argmax_{s_i \in S_i} E_{c_{-i}} max\{s_i, \sigma^*(c_{-i})\} - c_i.s_i$$

     for all $c_i$ and all $i$.
   - payoff from choosing $s_i^*(c_i) = 1$ is $1 - c_i$and the payoff from choosing $s_i^*(c_i) = 0$ is $p(s_{-i}^*(c_{-i})) \times 1 + (1 - p(s_{-i}^*(c_{-i})) \times 0 = p(s_{-i}^*(c_{-i})$.
   - Thus, the payoff from $s_i = 1$ is decreasing in $c_i = 1$ and the payoff of $s_i$ is independent of $c_i$.

## Example (Public goods provision cont.)

- Hence, look at monotonic cutoff strategies of the form

$$s_i(c_i) = \begin{cases} 1 & \text{if } c_i \leq c^* \\ 0 & \text{if } c_i > c^* \end{cases} \tag{15}$$

- Type $c^*$ of player $i$ must be indifferent between contributing and not, so

$$1 - c^* = p(s^*_{-i}(c_{-i}) = 1) = p(c_{-i} \leq c^*) = \frac{c^*}{2}$$

  or $c^* = \frac{2}{3}$. Where, remember from the *i.i.d* and uniform distribution of types that,

$$\int_0^{c^*} \frac{1}{2} d\theta_i = \frac{c^*}{2}$$

- Therefore, all players with private cost below $\frac{2}{3}$ contribute, while players with $c_i > \frac{2}{3}$ do not.

### Definition (Bayesian equilibrium with Mixed Strategie)

A Bayesian equilibrium with Mixed Strategies of a Bayesian game $[I, \{\Delta(S_i)\}, \{u_i(.)\}, \Theta, F(.)]$ is a mixed strategy profiles $\sigma = (\sigma_i, \sigma_{-i})$, such that for every player $i$ and every type $\theta \in \Theta_i$, we have

$$\sigma_i(.|\theta) \in \mathrm{argmax}_{\sigma_i \in \Delta(S_i)} F(\theta_{-i}|\theta_i)\Sigma_{s \in S}[\Pi_{j \neq i}\sigma_j(s_j|\theta_j)]\sigma_i(s_i)u_i(s|\theta)$$

## Example (Battle of Sexes with mixed Strategies)

- Battle of Sexes with incomplete information

P2 type $l$

| P1 | | B | S |
|---|---|---|---|
| | B | 2, 1 | 0, 0 |
| | S | 0, 0 | 1, 2 |

P2 type $h$

| P1 | | B | S |
|---|---|---|---|
| | B | 2, 0 | 0, 2 |
| | S | 0, 1 | 1, 0 |

- in the game type $l$ has two pyre Nash equilibria, while the type $h$ has no pure equilibrium
- We need to mix among the strategies
- $I = \{1, 2\}$, $S_1 = S_1 = \{A, S\}$
- $\Theta_1 = \{x\}$, $\Theta = \{l, h\}$
- $F_1(l|x) = F_1(h|x) = 1/2$, $F_2(x|l) = F_2(x|h) = 1$
- Player 1 mixes with probability $\sigma_1(B|x)$ and $1 - \sigma_1(B|x)$ between $B$ and $S$, respectively.

Example (Battle of Sexes with mixed Strategies, Cont.)

- If player 2's type is $h$, he mixes with probability $\sigma_2(B|l)$ and $1 - \sigma_2(B|l)$ between $B$ and $S$
- If player 2's type is $h$, he mixes with probability $\sigma_2(B|h)$ and $1 - \sigma_2(B|h)$ between $B$ and $S$
- Expected utility of player 1 is:

## EXAMPLE (BATTLE OF SEXES WITH MIXED STRATEGIES, CONT.)

- Expected utility of player 1 for the above setting is:

$$U_1(\sigma, x) = F_1(l|x)\sigma_2(B|l)\sigma_1(B|x)u_1(B(x), B(l), l, h, x)$$
$$+F_1(l|x)\sigma_2(S|l)\sigma_1(B|x)u_1(B(x), S(l), l, h, x)$$
$$+F_1(h|x)\sigma_2(B|h)\sigma_1(B|x)u_1(B(x), B(h), l, h, x)$$
$$+F_1(h|x)\sigma_2(S|h)\sigma_1(B|x)u_1(B(x), S(h), l, h, x)$$

$$+F_1(l|x)\sigma_2(B|l)\sigma_1(S|x)u_1(S(x), B(l), l, h, x)$$
$$+F_1(l|x)\sigma_2(S|l)\sigma_1(S|x)u_1(S(x), S(l), l, h, x)$$
$$+F_1(h|x)\sigma_2(B|h)\sigma_1(S|x)u_1(S(x), B(h), l, h, x)$$
$$+F_1(h|x)\sigma_2(S|h)\sigma_1(S|x)u_1(S(x), S(h), l, h, x)$$

## Example (Battle of Sexes with mixed Strategies, Cont.)

- **Player 1's expected payoff:** Given player 2's strategy $\sigma_2(B|l)$ and $\sigma_2(B|h)$ , her expected payoff to:
  - action $B$ (of P1) is
    $$\frac{1}{2}\sigma_2(B(l))(2) + \frac{1}{2}\sigma_2(B(h))(2) = \sigma_2(B(l)) + \sigma_2(B(h))$$
  - action $S$ (of P1) is
    $$\frac{1}{2}(1-\sigma_2)(B(l)(2) + \frac{1}{2}(1-\sigma_2(B(h))(2))$$
    $$= 1 - \frac{\sigma_2(B(l)) + \sigma_2(B(h))}{2}$$
- Therefore, her best response is to play $B$ if $\sigma_2(B(l)) + \sigma_2(B(h)) > \frac{2}{3}$ and to play $S$ if $\sigma_2(B(l)) + \sigma_2(B(h)) < \frac{2}{3}$.
- Find P2's expected payoff and the best response function. His best response is to play $B$ if $\sigma_1(B) < \frac{1}{3}$.