Simultaneous Move Games

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Overview

A few preliminary notations

- In this chapter, we study simultaneous move games using their strategic and normal forms.
- Before giving a general framework for the notations, we start with a specific example
- Let assume a game with three players that each of the players has different set of action, (or strategies).
  1. $S_1 = \{A, B\} = \mathcal{A}_1$
  2. $S_2 = \{H, T\} = \mathcal{A}_2$
  3. $S_3 = \{R, L\} = \mathcal{A}_3$
- Then, strategy profiles of the game is

$$S = S_1 \times S_2 \times S_3 = \{(A, H, R), ..., (B, T, L)\}$$

- In a general form of game notations the profiles are denoted by $s = (s_i, s_{-i})$.
- Specifically for $i = 1$ our example of interest we have $s_1 = A$ and $s_{-1} = (H, R)$

Normal form representation and its elements

- We use $\Gamma_N = [I, \{S_i\}, \{u_i(.)\}]$ to represent the games with pure strategies.
- Set of players $I = \{1, 2, ..., I\}$
- A profile of pure strategies for for player i's rivals by $s_{-i} = (s_1 s_2 ... s_{i-1} s_{i+1} ... s_I)$
- A strategy profiles is $S = \Pi_{i=1}^{I} S_i$, whose typical element is $s = (s_i, s_{-i})$
- The Cartesian product of action sets of all players excluding i is shown as follows:

$$S_{-i} = S_1 \times S_2 \times ... \times S_{i-1} \times S_{i+1} \times ... \times S_I$$

  where $s_{-i} \in S_{-i}$ and $s_i \in S_i$, then $s = (s_i, s_{-i}) \in S$
- **Payoff function** $u_i(.) : S \to \mathbb{R}$

Dominant strategy

Example

- **Prisoner's dilemma** Two allegedly engaged men are suspected in a serious crime, prosecutors try to extract a confession from each man
- They are separately kept in prison
- Each of the prisoners is privately told that if he is only one to confess, then he will be rewarded with a light sentence, while the other remaining silent will serve for 10-year sentence.
- See rest of story in the following strategic form game:

		Prisoner 2	
		Don't Confess	Confess
Prisoner 1	Don't Confess	-2, -2	-10,-1
	Confess	-1, -10	-5, -5

Dominant strategy

Example

cont'

- What will the **outcome** of this game be?
- The outcome= (*Confess, Confess*), **why?**

- Regardless of what his opponent does, player i is strictly better off playing *Confess* rather than *Don't Confess*.
- **This is precisely what is meant by a strictly dominant strategy**.

Lesson: self-interested behavior in games may not lead to socially optimal outcomes, or, in economics jargon language the outcome is caused by **externality** in choice.

Dominant strategy

Definition

A strategy $s_i \in S_i$ is a strictly dominant strategy for player i if for **all** $s_i' \neq s_i$, we have

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$$

for all $s_{-i} \in S_{-i}$.

- A strictly dominant strategy for i uniquely maximizes her payoff for any strategy profile of all other players.
- If such a strategy exists, it is highly reasonable to expect a player to play it. In a sense, this is a consequence of a player's "rationality".

Dominated strategies

- What about if a strictly dominant strategy doesn't exist?

Example

- See the following game: You can easily convince yourself that there are no strictly dominant strategies here for either player.

player 1 \ player 2	a	b	c
A	5,5	0,10	3,4
B	3,0	2,2	4,5

- Notice that regardless of whether Player 1 plays *A* or *B*, Player 2 does strictly better by playing *b* rather than *a*.

Dominated strategies

- That is, a is **strictly dominated** by b.

Definition

A strategy $s_i \in S_i$ is a strictly dominated strategy for player i if for there **exists** a strategy $s_i' \neq s_i$ such that for all $s_{-i} \in S_{-i}$, we have

$$u_i(s_i', s_{-i}) > u_i(s_i, s_{-i})$$

In this case, we say that s_i' strictly dominates s_i.

- Using this terminology, we can restate the definition of strictly dominant: A strategy s_i is strictly dominant if it strictly dominates all other strategies.

- It is reasonable that a player will not play a strictly dominated strategy, a consequence of rationality, again.

Dominated strategies

Example

- i=Player 2 and $-i$=Player 1
- $S_i = \{a, b, c\}$ and $S_{-i} = \{A, B\}$
- $s_i = a$ and $s_i' = b$ and either $s_{-i} = A$ or $s_{-i} = B$

Weakly dominated strategies

Definition

A strategy $s_i \in S_i$ is a strictly dominated strategy for player i if for there **exists** a strategy $s_i' \neq s_i$ such that for all $s_{-i} \in S_{-i}$, we have

$$u_i(s_i', s_{-i}) \geq u_i(s_i, s_{-i})$$

with strictly inequality for for s_i. In this case, we say that s_i' *weakly dominates* s_i.

Example

Player 2

		L	R
	U	$1, -1$	$-1, 1$
Player 1	M	$-1, 1$	$1, -1$
Strictly dominated	D	$-2, 5$	$-3, 2$

Player 2

		L	R
Weakly dominated	U	$5, 1$	$4, 0$
Player 1	M	$6, 0$	$3, 1$
Weakly dominated	D	$6, 4$	$4, 4$

Iterated deletion of dominated strategies

Definition

A game is weakly-dominance solvable if iterated deletion of weakly dominated strategies results in a unique strategy profile.

Example

A modifies version of prisoners dilemma

		Prisoner 2	
		Don't Confess	Confess
Prisoner 1	Don't Confess	$0, -2$	$-10, -1$
	Confess	$-1, -10$	$-5, -5$

Iterated deletion of dominated strategies

Example

cont'

- Does player 1 have any dominated strategy? **NO**
- What about player 2? He **does** have! Regardless of what player1 wants to choose, player 2 prefers to confess.
- $u_2(DC, C) > u_2(DC, DC)$ and $u_2(C, C) > u_2(C, DC)$, Don't Confess still is a strictly dominated for P2.
- Once P1 eliminates DC as a possible action by P2, C is P1's unambiguously optimal action.
- Rationality of players is a common knowledge

Iterated deletion of dominated strategies

- The order of deletion does not affect the set of strategies that remain in the end.

- We can eliminate the strictly dominated strategies all at once or any sequence, always we will end up the same strategies, then the player 2 will end up with *C1*.

Table: Note: A game with 2 strictly dominated strategies, *C1* and *C2*.

		Player 2		
		C1	C2	C3
	R1	4, 3	5,1	6, 4
Player 1	R2	2, 1	3, 4	3, 6
	R3	3, 0	4,6	2, 8

Iterated deletion of dominated strategies

- A constant sum game, always for any strategy, $s_1 + s_2 = 1$, every buyer will buy from the nearest store. The stores can located their stores in 0, 1, 2, 3, or 4 of a four-mile distance.

- payoffs are market share of two firms that are planing to locate their branches

- You can find the Equilibrium Strategies by applying the *max min* principle as well.

Iterated deletion of dominated strategies

- Payoff Matrix for player 1

		Player 2				
		0	1	2	3	4
	0	4/8	1/8	2/8	3/8	4/8
	1	7/8	4/8	3/8	4/8	5/8
Player 1	2	6/8	5/8	4/8	5/8	6/8
	3	5/8	4/8	3/8	4/8	7/8
	4	4/8	3/8	2/8	1/8	4/8

Table: Note: In a constant sum game, one can find payoffs of the second player by $s_1 + s_2 = 1$. Optimal solution is (2, 2) and payoffs are (0.5, 0.5)

- R1 is dominated by R2
- R4 is dominated by R3

Iterated deletion of dominated strategies

- One must eliminate only the strictly dominated strategies.
- One cannot eliminate a strategy if it is weakly dominated but not strictly dominated.

Example

		Player 2	
		L	R
Player 1	T	1, 1	0,0
	B	0, 0	0, 0

- T does not strictly dominate B, and L does not strictly dominate R

Strictly dominant mixed strategies

- Are there always either a pure dominant or dominated strategies in a game? **NO**

Example

Table: Note: The game doesn't have any pure dominated strategy, but one can find mixed dominant strategies.

Player 2

		L	R
Player 1	U	10, 1	0, 4
	M	4, 2	4, 3
	D	0, 5	10, 2

- No matter which strategy the player 2 plays, player 1 prefers to mix U and D, but never play M, **why?**.
- Decide between U and D by flipping a coin, forget about strategy M

Strictly dominant mixed strategies

- Expected utility of player 1, if Player 2 plays L

  $$E(u_1(p,L)) = p\times 10 + 0\times 4 + (1-p)\times 0 > 0\times 10 + 1\times 4 + 0\times 0$$

  the inequality holds for all $p > 0.4$
- Expected utility of player 1, if Player 2 plays R

  $$E(u_1(p,R)) = p\times 0 + 0\times 4 + (1-p)\times 10 > 0\times 0 + 1\times 4 + 0\times 10$$

  the inequality hold for all $p < 0.6$.
- So, for $0.4 < p < 0.6$ the mixed strategy of $p\times U + (1-p)\times D > M$ [unconditional to what actions P2 takes], that is, randomizing over U and D is preferred to the degenerated lottery of M.
- Can we prove formally the simple idea? **Yes**
- At first we need to define the key concept of *Mixed Strategies*

Strictly dominant mixed strategies

Definition

a strategy $\sigma_i \in \Delta(S_i)$ is strictly dominated for player i in game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(.)\}]$, if there exists another strategy $\sigma_i' \in \Delta(S_i)$ such that for all $\sigma_{-i} \in \Pi_{j \neq i} \Delta(S_i)$,

$$u_i(\sigma_i', \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i}).$$

In this case, we say that strategy σ_i' strictly dominates strategy σ_i.

Example

In the above preliminary example the $-i$ = *Player 2* and we took two mixed strategies over L and R. So, the σ_{-i} can take the following values in turn.

1. $q = 1$ for L and $1 - q = 0$ for R
2. $q = 0$ for L and $1 - q = 1$ for R

We can restrict σ'_{m1} to the values listed in below.

1. $0.4 < \sigma'_{11} < 0.6$, $\sigma'_{21} = 0$ and $0.4 < \sigma'_{31} < 0.6$ with $\sigma'_{11} + \sigma'_{21} + \sigma'_{31} = 1$ for randomizing over U, M and D
2. We used $\sigma_{11} = 0$, $\sigma_{21} = 1$ and $\sigma_{31} = 0$ as one alternative for probability values for randomizing over U, M and D under σ_1.

Strictly dominant mixed strategies

Example

A technical example for equation (3 in below): Suppose that three people are involved in game with M strategies for each on. Then the vNM utility function for player one is:

$$\sum_{s_1 \in S_1} \sum_{s_{-1} \in S_{-1}} [\Pi_{j \neq 1} \sigma_j(s_j)] \times \sigma_1(s_1).u_1(s_1, s_{-1})$$

$$= \sum_i \sum_j \sum_k p_i \, q_j \, z_k \, u_1(s_{i1}, s_{j2}, s_{k3})$$

$$= \sum_j \sum_k q_j \, z_k \sum_i p_i \, u_1(s_{i1}, s_{j2}, s_{k3})$$

$$= \sum_j \sum_k q_j \, z_k \, u_1(p, s_{j2}, s_{k3}) = \sum_j \sum_k q_j \, z_k \, u_1(\sigma_1, s_{-1})$$

Where, p_i, q_j and z_k are probabilities.

Strictly dominant mixed strategies

Theorem

Player i's mixed strategy σ_i *is strictly dominated in game* $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(.)\}]$ *if only if there exists another* $\sigma_i' \in \Delta(S_i)$ *such that*

$$u_i(\sigma_i', s_{-i}) > u_i(\sigma_i, s_{-i}).$$

for all $s_{-i} \in S_{-i}$.

Proof.

by definition the expected utility is:

$$u_i(\sigma_1, \sigma_2, ..., \sigma_I) = \sum_{s \in S} [\sigma_1(s_1) \times \sigma_2(s_2) \times ... \times \sigma_I(s_I)].u_i(s)$$

Which is representable by:

$$u_i(\sigma_i, \sigma_{-i}) = \sum_{s_i \in S_i} \sum_{s_{-i} \in S_{-i}} [\Pi_{j \neq i} \sigma_j(s_j) \times \sigma_i(s_i)].u_i(s_i, s_{-i}). \quad (1)$$

Strictly dominant mixed strategies

- For the above formula to make sense, it is critical that each player is randomizing independently. That is, each player is independently tossing her own die to decide on which pure strategy to play.

Proof.

And in the same way the expected utility for player i, once randomizing by $\sigma' i$, is:

$$u_i(\sigma'_i, \sigma_{-i}) = \sum_{s_i \in S_i} \sum_{s_{-i} \in S_{-i}} [\Pi_{j \neq i} \sigma_j(s_j) \times \sigma'_i(s_i)].u_i(s_i, s_{-i}). \quad (2)$$

$\square$

Strictly dominant mixed strategies

Proof.

Subtracting (1) from (2)

$$u_i(\sigma_i', \sigma_{-i}) - u_i(\sigma_i, \sigma_{-i}) =$$

$$\sum_{s_i \in S_i} \sum_{s_{-i} \in S_{-i}} [\Pi_{j \neq i} \sigma_j(s_j) \times \sigma_i'(s_i)].u_i(s_i, s_{-i}) -$$

$$\sum_{s_i \in S_i} \sum_{s_{-i} \in S_{-i}} [\Pi_{j \neq i} \sigma_j(s_j) \times \sigma_i(s_i)].u_i(s_i, s_{-i})$$

and applying $\sum_{s_i \in S_i}$ on the right hand side, over $\sigma_i(s_i)].u_i(s_i, s_{-i})$ and $\sigma_i'(s_i)].u_i(s_i, s_{-i})$ gives rise to:

$$u_i(\sigma_i', \sigma_{-i}) - u_i(\sigma_i, \sigma_{-i}) =$$

$$\sum_{s_{-i} \in S_{-i}} [\Pi_{j \neq i} \sigma_j(s_j)].u_i(\sigma_i', s_{-i}) - \sum_{s_{-i} \in S_{-i}} [\Pi_{j \neq i} \sigma_j(s_j)].u_i(\sigma_i, s_{-i}) \quad (3)$$

□

Strictly dominant mixed strategies

Proof.

After a few manipulation it follows that:

$$u_i(\sigma'_i, \sigma_{-i}) - u_i(\sigma_i, \sigma_{-i}) =$$

$$\sum_{s_{-i} \in S_{-i}} [\Pi_{j \neq i} \sigma_j(s_j)].[u_i(\sigma'_i, s_{-i}) - .u_i(\sigma_i, s_{-i})].$$

Therefore, $u_i(\sigma'_i, \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i})$ for all σ_{-i}, if only if, $u_i(\sigma'_i, s_{-i}) > .u_i(\sigma_i, s_{-i})$ for all s_{-i}. □

- What do we learn from this theorem?
- As our the simple example showed, keep the strategies of your rival pure, while you are randomizing over your own actions.

Strictly dominant mixed strategies

Theorem

Player i's pure strategy s_i is strictly dominated in game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(.)\}]$ if only if there exists another $\sigma_i' \in \Delta(S_i)$ such that

$$u_i(\sigma_i', s_{-i}) > u_i(s_i, s_{-i}).$$

for all $s_{-i} \in S_{-i}$.

Proof.

This is a corollary of the just proved theorem. Only assume that σ is a degenerated lottery, then the conclusion follows, however, more complicated proof exists as well. $\square$

Example

- Players' common knowledge of each others' rationality and the game structure allows us to eliminates more that strictly dominated strategies.
- Consider the 2-player game which does not have any pure strictly dominated strategy, **but it has mixed one**. Find it.

		Player 2			
		b1	b2	b3	b4
	a1	0, $\underline{7}$	2, 5	$\underline{7}$, 0	0, 1
	a2	5, 2	$\underline{3}$, $\underline{3}$	5, 2	0, 1
Player 1	a3	$\underline{7}$, 0	2, 5	0, $\underline{7}$	0, 1
	a4	0, $\underline{0}$	0, 2	0, $\underline{0}$	$\underline{10}$, -1

Rationalizable Strategies

Example

cont.

- b_4 is a strictly dominated by mixed of b_1 and b_3, just take $\sigma_1 = 0.5$
- Then, by mixing a_1 and a_3 the action a_4 can be eliminated
- A chain of justifications $(a_1, b_3, a_3, b_1, a_1, b_3, ...)$
- Which strategy the player 1 will play if her rival b_1? P1 will play a_3, because it gives the highest payoff. It is the best response that P1 can do.
- P1 can construct an infinite chain of justification for playing a_2 by the belief that P2 will play $b_2, (a_2, b_2, a_2, b_2, ...)$

Nash Equilibrium

- One of the most well-known solution concepts in game theory, pure (or Mixed) Nash equilibrium

Example

Let start with the historical idea of Cournot duopoly equilibrium.

- Two profit maximizer firms with homogeneous products
- Aggregate market demand is $p = a - b(q_1 + q_2)$ and their total cost are cq_1 and cq_2.
- The objective function for firm 1 is: $max_{q_1}\ \pi_1(q_1, q_2) = [a - b(q_1 + q_2)]q_1 - cq_1$, s.t. $q_2 = \bar{q}_2$
- For firm 2 is: $max_{q_2}\ \pi_2(q_2, q_2) = [a - b(q_1 + q_2)]q_2 - cq_2$, s.t. $q_1 = \bar{q}_1$

Nash Equilibrium

Example

The first order condition of profit maximization for firm 1:

$$a - 2bq_1 - b\,\bar{q}_2 - c = 0, \qquad q_1^* = \frac{a - bq_2^* - c}{2b} \qquad (4)$$

and that for firm 2 is:

$$a - 2bq_2 - b\,\bar{q}_1 - c = 0, \qquad q_2^* = \frac{a - bq_1^* - c}{2b} \qquad (5)$$

$$q_1^* = q_2^* = (a - c)/3b$$

Functions (4) and (5) are called reaction functions. What do we learn from these solutions?

- Firm 1 considers the level of Firm 2's production, while it is deciding about its q_1

Nash Equilibrium

- Unlike with our earlier solution concepts (dominance), Nash equilibrium applies to a profile of strategies rather than any individual's strategy.

Definition

A strategy profile $(s_1^*, ..., s_I^*) = (s_i^*, s_{-i}^*) \in S$ is a Pure Strategy Nash Equilibrium if for all i and all $s_i' \in S_i$

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i', s_{-i}^*).$$

- In a Nash equilibrium, each player's strategy must be a best response to those strategies of his opponents that are components of the equilibrium.
- **Remark:** Every finite game of *perfect information* has at least a pure strategy of Nash equilibrium

Nash Equilibrium

- If the $-i$ players play their Nash Equilibrium Strategy s^*_{-i}, then player i has no incentive to unilaterally deviate from s_i^* to any $s_i' \in S_i$. Because, he cannot improve his payoff.

- So, given s^*_{-i} for players $-i$, the $u_i(s_i^*, s^*_{-i})$ gives the highest payoff for i.

- **Remark:** Every finite game of *perfect information* has a pure strategy Nash equilibrium

Nash Equilibrium

Example

- Remember the 2-player game with no pure strictly dominated strategy, developed by Bernheim (1984).

		Player 2			
		b1	b2	b3	b4
	a1	0, $\underline{7}$	2, 5	$\underline{7}$, 0	0, 1
	a2	5, 2	$\underline{3}$, $\underline{3}$	5, 2	0, 1
Player 1	a3	$\underline{7}$, 0	2, 5	0, $\underline{7}$	0, 1
	a4	0, $\underline{0}$	0, 2	0, $\underline{0}$	$\underline{10}$, -1

- P1 can construct an infinite chain of justification for playing *a2* by the belief that P2 will play *b2*, (*a2*, *b2*, *a2*, *b2*,...)
- $s_1^* = a_2$ and $s_2^* = b_2$ is a pure Nash equilibrium of the game, and the best responses is $a_2 = BR_1(b_2)$.

Best-response correspondence

- Player i's BR correspondence $b_i : S_{-i} \rightarrow S_i$, is the correspondence that assigns to each $s_{-i} \in S_{-i}$ the set

$$b_i(s_{-i}) = \{s_i \in S_i : u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}), \forall s_i' \in S_i\}$$

- Why correspondence and not function? see the response of P2 to a4 of player 1

- with this notation, we can restate the definition the Nash Equilibrium: The strategy profile (s_i, s_{-i}) is a Nash equilibrium of a game with pure strategies if and only if $s_i = b_i(s_{-i})$ for all $i \in I$.

Mixed Strategy Nash Equilibria

Example

Meeting in New York game, Grand Central or Empire State, 1 mile away.

Table: A game with more than one Nash Equilibrium strategy

		Mr. Schelling: Empire State	Mr. Schelling: Grand Central
Mr. Thomas	Empire State	100, 100	0, 0
	Grand Central	0, 0	100, 100

- Which of the profiles is the solution for the game? Both of them, and there is one more with mixing.
- Recall the standard form of matching the pennies **Version (A)**, there was no pure strategy equilibrium.

Mixed Strategy Nash Equilibria

- What is a mixed Nash Equilibrium?

Definition

A mixed strategy profile $\sigma = (\sigma_1, ..., \sigma_I)$ (or (σ_i, σ_{-i})) constitutes a Nash equilibrium of game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(.)\}]$ if for every $i = 1, ..., I$,

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})$$

for all $\sigma_i' \in \Delta(S_i)$.

Example

Table: A game with mixed strategy

Palyer B

		l q	r 1-q
Player A	u p	a_{ul}, b_{ul}	a_{ur}, b_{ur}
	d 1-p	a_{dl}, b_{dl}	a_{dr}, b_{dr}

Mixed Strategy Nash Equilibria: Example

Example

Existence of Mixed Strategy Nash Equilibria

- The a and b in each profile are utility of player A and B, respectively
- What are the reaction function of player One and Two, when they are randomizing over their actions?

$$u_A(p,q) = p[q.a_{ul} + (1-q)a_{ur}] + (1-p)[qa_{dl} + (1-q)a_{dr}]$$

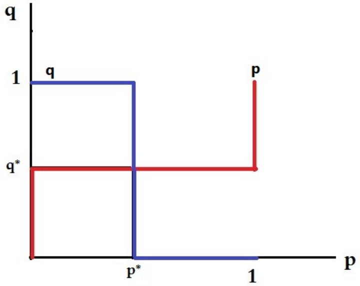
- Suppose that $q.a_{ul} + (1-q)a_{ur} > qa_{dl} + (1-q)a_{dr}$, implying that $q > (a_{dr} - a_{ur})/(a_{ul} - a_{ur} - a_{dl} + a_{dr})$, then player A will choose to play u with $p = 1$.
- He will play u with $1 - p = 1$, when the reverse is true for q.because a larger weight for $qa_{dl} + (1-q)a_{dr}$ gives a higher level of payoff fort A.

Mixed Strategy Nash Equilibria: Example

- He is indifferent between u and d for $q^* = (a_{dr} - a_{ur})/(a_{ul} - a_{ur} - a_{dl} + a_{dr})$.
- making use of the vNM utility function for player B:

$$u_B(p, q) = q[p.b_{ul} + (1-p)b_{ur}] + (1-q)[pb_{dl} + (1-p).b_{dr}]$$

  and following the similar steps one can find p^*.
- the single mixed Nash equilibrium of the game is (p^*, q^*)

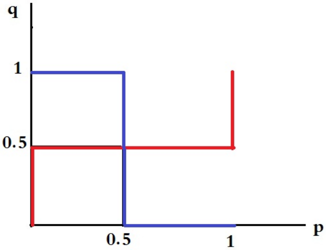


Example

Palyer 2

		H q	T 1-q
	H p	1, -1	-1, 1
Player 1	T 1-p	-1, 1	1, -1

- vNM utility for player 1 is$u_1(p, q) =$ $p[q.1 + (1-q)(-1)] + (1-p)[q(-1) + (1-q)1]$.
- She will play H with probability 1 if $q > 0.5$, because, $2q - 1 > 1 - 2q$. Then $q > 0.5$



Example

- The following game has two pure and one mixed Nash equilibria.

		B	
		L (q)	R (1-q)
A	U (p)	1, -1	3, 0
	D (1-p)	4, 2	0, -1

Figure: Two pure Nash Equilibria, (U, R) and (D, L), **both with probability 1**.

- vNM utility function for Player A

$$u_1(p, q) = p(q + 3(1 - q)) + (1 - p)(4q + (1 - q) \times 0)$$

$$u_1(p, q) = p(3 - 2q) + (1 - p)(4q)$$

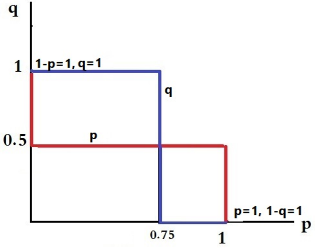
- vNM utility function for Player B

$$u_2(p,q) = q(p(-1) + (1-p)(2)) + (1-q)((1-p)(-1))$$

- For what level of p, player B will choose action L?

$$p(-1) + (1-p)(2) > (1-p)(-1)$$
$$\Rightarrow p < \frac{3}{4}$$

- if $p < \frac{3}{4}$ then she will play L with $q = 1$



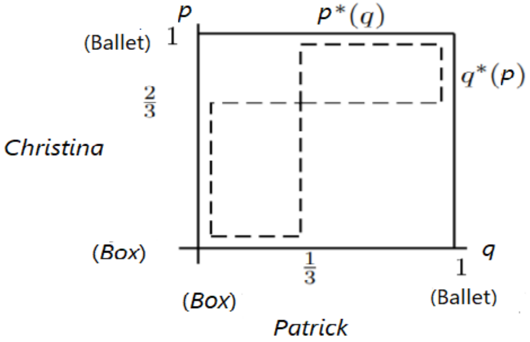
Example

- A husband and a wife are deciding to go for watching *ballet* or *Box*.
- They both would rather spend the evening together than apart, but **Patrick** would rather be together at the *Box* while **Christina** would rather be together at the *ballet*.
- The payoff matrix for the spouses is:

		Patrick	
		ballet q	*Box* (1-q)
Christina	*ballet* p	2, 1	0, 0
	Box (1-p)	0, 0	1, 2

Example

- There two pure Nash equilibria (*ballet, ballet*) and (*Box, Box*), and a mixed Nash equilibrium $(p^* = 2/3, \quad q^* = 1/3)$.



Example

cont.

$$p = \begin{cases} 0 & \text{if } q < 1/3 \text{ i.e Christina will choose Box} \\ [0,1] & \text{if } q = 1/3 \text{ i.e Christina is indiferent between ballet and Box} \\ 1 & \text{if } q > 1/3 \text{ i.e Christina will choose ballet} \end{cases} \tag{1}$$

$$q = \begin{cases} 0 & \text{if } p < 2/3 \text{ i.e Patrick will choose Box} \\ [0,1] & \text{if } p = 2/3 \text{ i.e Patrick is indiferent between ballet and Box} \\ 1 & \text{if } p > 2/3 \text{ i.e Patrick will choose ballet} \end{cases} \tag{2}$$

What does it mean to play Mixed Strategy Nash Equilibria?

- Different interpretations
  1. Randomize to confuse your opponent, games of Matching pennies and penalty kick in football
  2. Randomize when uncertain about the other's action, Battle of the Sexes and meeting in New York city
  3. Mixed strategies are a concise description of what might happen in repeated play. Count of pure strategies in the limit.
  4. Mixed strategies describe population dynamics. Two agents are chosen from a big population all having deterministic strategies. Mixed strategy gives the probability of getting each pure strategy.

Example

- So far we analyzed strictly dominated strategies with only two players.
- What if we have three players?.

Lance chooses *steroids*

		Floyd	
		Steroids	*No steroids*
Bernhard	*Steroids*	2,3,3	3,1,5
	No steroids	1,4,5	5,2,6

Lance chooses *no steroids*

		Floyd	
		Steroids	*No steroids*
Bernhard	*Steroids*	3,4,1	4,2,2
	No steroids	5,5,2	6,6,4

- First, we check if Lance (the matrix player) has some strictly dominated strategy.

Example

- We compare $u_3(steroids, s_1, s_2)$ against $u_3(No\ \ steroids, s_1, s_2)$ where s_1 and s_2 are fixed across matrices.
- No steroids is a strictly dominated strategy for Lance, as it yields a lower payoff than steroids, for every profile (s_1, s_2) of the other two athletes.

Lance chooses *steroids*

Bernhard \ Floyd	*Steroids*	*No steroids*
Steroids	2,3,3	3,1,5
No steroids	1,4,5	5,2,6

Lance chooses *no steroids*

Bernhard \ Floyd	*Steroids*	*No steroids*
Steroids	3,4,1	4,2,2
No steroids	5,5,2	6,6,4

Example

- Then, we can delete "*No steroids*" from Lance by deleting the right hand matrix.
- When Lance will use *Steroids*. Bernhard Can Now Deduce That Floyd's Dominant Strategy Is to Use Steroids

		Floyd	
		Steroids	No steroids
Bernhard	Steroids	2,3,3	3,1,5
	No steroids	1,4,5	5,2,6

EXAMPLE

- Hence, the above matrix reduces to the following 2×1:

		Steroids
Bernhard	Steroids	2,3,3
	No steroids	1,4,5

- Moving now to Bernhard (row player), we note that "*No steroids*" is strictly dominated by "*steroids*".
- Hence, the only strategy profile surviving Iterated Deletion of Strongly Dominated Strategies **IDSDS** is (*Steroids*, *Steroids*, *Steroids*)

- What is the action set of each player in the following 3-person game?
- Write explicitly the best response of player 3 to action profile of the other players.
- Find Nash equilibria of the game.

$$\begin{array}{ccc} & j=1 & j=2 \\ \begin{matrix} i=1 \\ i=2 \end{matrix} & \multicolumn{2}{c}{\begin{bmatrix} (4,3,2) & (3,4,4) \\ (3,4,3) & (4,3,5) \end{bmatrix}} \\ & \multicolumn{2}{c}{k=1} \end{array} \qquad \begin{array}{ccc} & j=1 & j=2 \\ \begin{matrix} i=1 \\ i=2 \end{matrix} & \multicolumn{2}{c}{\begin{bmatrix} (2,1,6) & (0,0,8) \\ (0,0,7) & (1,2,9) \end{bmatrix}} \\ & \multicolumn{2}{c}{k=2} \end{array}$$

Hint.

- Use the following rule
  $s_i = BR_i(s_{-i})$ or $s_3 = BR_3(s_1, s_2)$
- **Solution:** $(s_1^*, s_2^*, s_3^*) = (1, 1, 2)$ and $(s_1^*, s_2^*, s_3^*) = (2, 2, 2)$ are pure Nash equilibria.