Examples for Chapter 6

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Overview

Lotteries for continuous outcomes

- Example 1. Suppose that probability distribution (lottery 1) $F_1(x)$ is of the form of

$$F_1(x) = \int (1/2) dx$$

for $x \in [1, 3]$, and the lottery two $F_2(x)$ has the following form

$$F_2(x) = \int (1/3) dx$$

for $x \in [1, 4]$. Then, lottery $F_1(x)$ is at least as good as lottery $F_2(x)$ if only if

$$\int u(x) dF_1(x) \geq \int u(x) dF_2(x)$$

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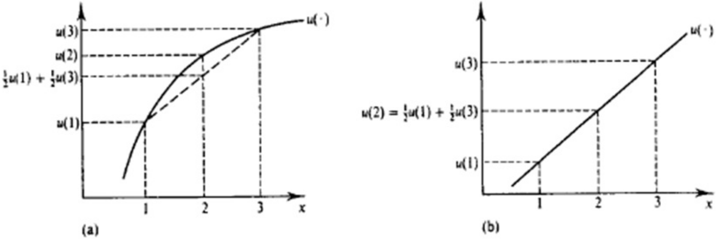
Attitude toward risk: Risk aversion

- The expected value of x, in our example wealth, is a degenerated lottery $\int x dF(x)$ with $p = 1$



Attitude toward risk: Risk aversion

- Locus of the $p.u(x_1) + (1-p).u(x_2)$ depends on the value of p.

- Expected value of utility shows the value of gamble for the agent.
- For a risk averse agent, the expected value is less that utility of the degenerated (a certain value of wealth) lottery.

Definition

Risk Aversion

$$\int u(x)dF(x) \leq u(\int xdF(x))$$

Risk aversion and actuarially un-fair pricing

- Suppose that insurance policy pricing is not actuarially fair, show that a risk averse agent does not insure whole risk.

$$Max\ \pi u(w - D - \alpha q + \alpha) + (1 - \pi)u(w - \alpha q)$$
$$F.O.C : \pi(1-q)u'(w-D-\alpha q+\alpha)-(1-\pi)qu'(w-\alpha q) \leq 0$$

- recall the Kuhn Tucker necessary condition in mathematical programming.
$$\Rightarrow \pi(1 - q)u'(w - D - \alpha q + \alpha) = (1 - \pi)qu'(w - \alpha q)$$

$$q \geq \pi \Rightarrow (1 - \pi) \geq 1 - q \Rightarrow q(1 - \pi) \geq \pi(1 - q)$$
$$\Rightarrow u'(w - D - \alpha q + \alpha) \geq u'(w - \alpha q)$$

- Note that the agent is risk averse, namely $u''(.) \leq 0$, so we will have: $\Rightarrow w - D - \alpha q + \alpha \leq w - \alpha q \Rightarrow \alpha \leq D$

Risk aversion and attitude towards risk

- Certainty equivalent of a lottery is a value of $c(F,u) = x$ which its utility is equal to the expected value of the lottery. In other word, certainty equivalent of a lottery is the value that an agent is willing to get it and leave the game or gamble, $u(c(F,u)) = \int u(x)dF(x)$

- Value of a game is evaluated by expected value of the game: $\int u(x)dF(x)$
- An agent is called risk averse if,

$$c(F,u) \leq \int x dF(x)$$

.

Certainty equivalent: Example

- Let probability density function for a risky asset be $f(x) = (1/2)$, where, $x \in [1, 3]$. The agent's bernoulli utility function defined on x is assumed as, $u(x) = x^{1/2}$. Show that the consumer is risk averse.
- sketch solution: (a) find the expected utility function, (b) find the value of x which equates utility level with the expected value, (c) find the expected value of x. Now compare the (b) and (c).
  1. $E(u(x)) = \int_1^3 (x^{1/2}/2)dx = 1.4$
  2. $u[c(F, u)] = x^{1/2} = 1.4$ which gives $c(F, u) = (1.4)^2 = 1.96$
  3. $E(x) = \int_1^3 (x/2)dx$=2
  4. $1.6 = c(F, u) \leq E(x) = 2$
  5. Therefore, the agent is risk averse, or, the bernoulli utility function is Concave.

Probability premium value and risk aversion: Example

Definition

Probability premium

$$u(x) = u(x - \epsilon)(0.5 - \pi(.)) + u(x + \epsilon)(0.5 + \pi(.))$$

- Take $x = 4$, $\epsilon = 1$ and $u(x) = \sqrt{x}$.
- for the given values of x, ϵ and bernoulli utility function, show that Probability premium is positive. Why is this so?
- Solution:

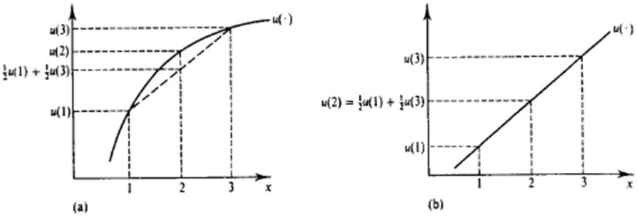
$$u(4) = u(4 - 1)(0.5 - \pi(.)) + u(4 + 1)(0.5 + \pi(.))$$

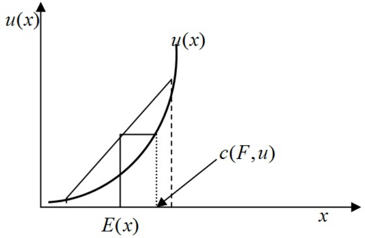
$$\sqrt{4} = \sqrt{4 - 1}(0.5 - \pi(.)) + \sqrt{4 + 1}(0.5 + \pi(.))$$

$$\pi(.) = 0.0357$$

- Change the utility from to $u(x) = x^2$ and compare the result, is that positive yet?

Attitude towards risk: From risk aversion to risk lover





- 1. preference of a risk averse decision maker, 2. risk neutral, and preference of a risk lover decision maker

How to measure the risk aversion

- The utility functions differ in terms of their curvature
- Can we use this property as a measure of risk aversion? YES
- Arrow and Pratt have introduced the Absolute Risk Aversion Coefficient

Definition

Coefficient of Absolute Risk Aversion: The Arrow-Pratt coefficient of absolute risk aversion at x is defined as:

$$r_A(x) = -\frac{u''(x)}{u'(x)}$$

Note: we are dividing the $u''(x)$ by $u'(x)$ to make it invariant to any linear increasing transformation, compare $r_A(x)$ for $u(x) = \sqrt{x}$ and $u(x) = \alpha\sqrt{x}$.

Interpersonal risk aversion comparison

- Given two individuals with bernoulli utility function, $u_1(x)$ and $u_2(x)$, how can one compare their risk aversion intensity?

- There are many ways:

  1. concavity of their utility function
  2. certainty equivalent value comparison
  3. probability premium values
  4. Arrow-Pratt coefficient of absolute risk aversion

Curvature of bernoulli utility functions and the values of $c(F, u)$

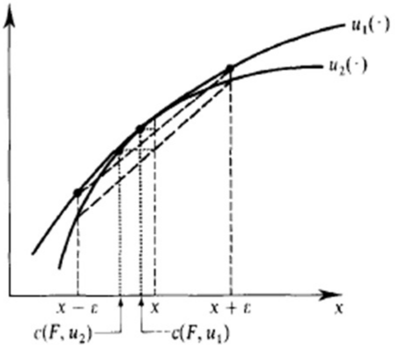


Figure: The utility function with greater curvature gives smaller value for $c(F, u)$

Comparisons across individuals

- the following statements are equivalent
  - $r_A(x, u_2) \geq r_A(x, u_1)$ for every x
  - There is an increasing concave function ψ such that $u_2(x) = \psi(u_1(x))$ at all x, that is $u_2(x)$ is more concave than $u_1(x)$, therefore, former is more risk averse than the later .
  - $c(F, u_1) \geq c(F, u_2)$
  - $\pi(x, \epsilon, u_2) \geq \pi(x, \epsilon, u_1)$
- Example: $u_1(x) = \sqrt{x}$ and $u_2(x) = (\sqrt{x})^{3/4}$

Comparisons across individuals

Theorem

If $r_A(x, u_2) \geq r_A(x, u_1)$ *for every* x*, then there is an increasing concave function* ψ *such that* $u_2(x) = \psi(u_1(x))$ *at all* x *and* $u_2(x)$ *is more risk averse than* $u_2(x)$*.*

Proof:

- $u_2'(x) = \psi'(u_1(x))u_1'(x)$
- $u_2''(x) = \psi''(u_1(x))(u_1'(x))^2 + \psi'(u_1(x))u_1''(x)$
- $-\frac{u_2''(x)}{u_2'(x)} = -\frac{\psi''(u_1(x))(u_1'(x))^2 + \psi'(u_1(x))u_1''(x)}{\psi'(u_1(x))u_1'(x)}$
- $r_A(x, u_2) = -\frac{\psi''(u_1(x))(u_1'(x))}{\psi'(u_1(x))} + r_A(x, u_1)$
- $-\frac{\psi''(u_1(x))(u_1'(x))}{\psi'(u_1(x))} \geq 0$

Comparisons across individuals:Example

Example

Suppose that the utility function of individual 2 is concave transformation of individual 1, as $u_1(x) = \sqrt{x}$ and $u_2(x) = (\sqrt{x})^{3/4}$. Show that $r_A(x, u_2) \geq r_A(x, u_1)$

Solution

- $r_A(x, u_1) = \frac{1}{2}\frac{1}{x}$
- $r_A(x, u_2) = \frac{5}{8}\frac{1}{x}$

Payoff distributions comparison in terms of return and risk

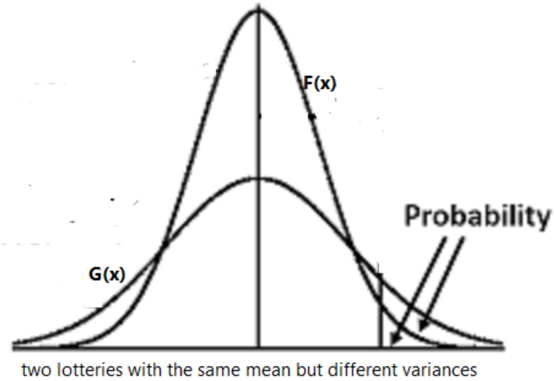


Figure: Two lotteries with the same means but different variances

Payoff distributions comparison in terms of return and risk

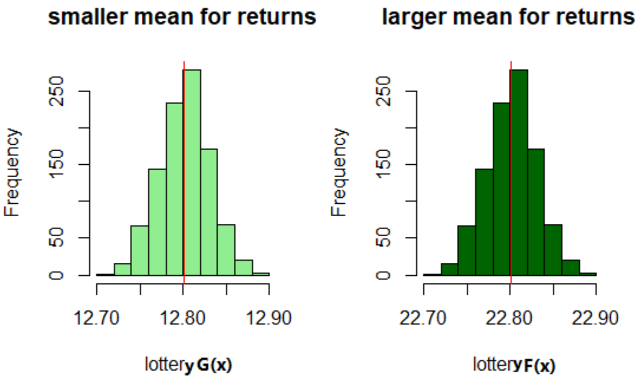


Figure: Two lotteries with the same variances but different means

Graphical representation of First order stochastic dominance

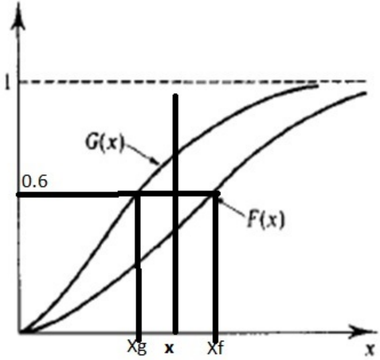


Figure: $G(.)$ and $F(.)$ are probability distributions. For every given level of probability $[F(.)$ and $G(.)]$, return of lottery $F(.)$ dominates $G(.)$

First Order Stochastic Dominance

Definition

First order stochastic dominance The lottery (distribution) $F(.)$ first order stochastically dominates lottery $G(.)$ if, for every nondecreasing function $u : \mathbf{R} \to \mathbf{R}$, we have

$$\int u(x)dF(x) \geq \int u(x)dG(x)$$

.

First Order Stochastic Dominance

Theorem

First order stochastic dominance: *The lottery (distribution) of monetary payoffs* $F(.)$ *first-order stochastically dominates lottery* $G(.)$ *if only if* $F(.) \leq G(.)$ *for every* x.

First Order Stochastic Dominance: Proof

Proof.

The *only if part* [$\int u(x)dF(x) \geq \int u(x)dG(x)$ **only if** $F(.) \leq G(.)$ for every x. [A only if B $\equiv$ if A then B]

- or equivalently, if $\int u(x)dF(x) \geq \int u(x)dG(x)$, then $F(.) \leq G(.)$ for every x .]
- We apply the contour positive reasoning method [if $\neg B$ then $(\neg A)$] to prove the statement.
- Specifically, if $\neg B$ $\{F(.) > G(.)\}$, then $\neg A$ $\{\int u(x)dF(x) < \int u(x)dG(x)]\}$.
- By $[\neg B]$, we have $H(x) = F(x) - G(x) > 0$, and we want to show that $\int u(x)dF(x) - \int u(x)dG(x)] < 0$.

$\square$

First order Stochastic dominance

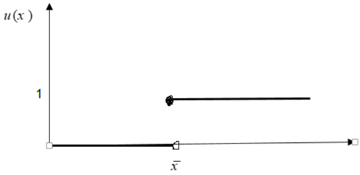


Figure: the step utility function $u(x) = 0$ for $x < (\bar{x})$ and $u(x) = 1$ for $x \geq (\bar{x})$

- the step utility function has the property that $\int u(x)dH(x) = \int_{-\infty}^{\bar{x}} u(x)dH(x) + \int_{\bar{x}}^{+\infty} u(x)dH(x)$.

First order Stochastic dominance

Proof.

- the first part of the integral equals zero and for the second part we have $H(\infty) - H(\bar{x}) = -H(\bar{x}) < 0$, since $H(\infty) = 0$
- It gives $\int u(x)dF(x) - \int u(x)dG(x)] = -[F(\bar{x}) - G(\bar{x})] = [G(\bar{x}) - F(\bar{x})] < 0$ is satisfied for every $\bar{x}$.
- Since $G(\bar{x}) < F(\bar{x})$, we conclude that $\neg A$ is true .
  Q.E.D

$\square$

First order Stochastic dominance, the IF part

Proof.
The *if part* $[\int u(x)dF(x) \geq \int u(x)dG(x)$ **if** $F(.) \leq G(.)$ for every x. [A if B $\equiv$ if B then A]. We use a direct method to prove the statement.

- if $F(.) \leq G(.)$ then $[\int u(x)dF(x) \geq \int u(x)dG(x)$
- Let construct $H(x) = F(x) - G(x) \leq 0$ and suppose $u(x) = u$ and $dH(x) = dv$.
- Then by *integrating by part* we have:

$$\int u(x)dH(x) = [u(x)H(x)]_0^\infty - \int u'(x)H(x)dx$$

□

First order Stochastic dominance, the IF part

- The first part of the integral equals zero $[H(0) = H(\infty) = 0]$ and for the second part we have $-\int u'(x)H(x)dx$
- From risk aversion assumption we have $u'(x) \geq 0$, and we know from the definition of $H(x)$ that, it must be non-positive, therefore:

$$\int u(x)dH(x) = \int u(x)dF(x) - \int u(x)dG(x)$$

$$= -\int u'(x)H(x)dx \geq 0$$

Q.E.D

Graphical representation of Second order stochastic dominance

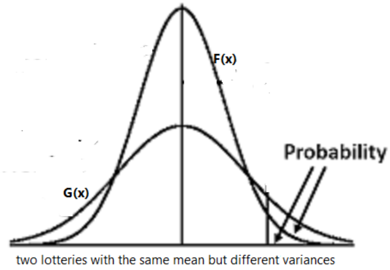


Figure: Density distribution functions for lotteries $F(.)$ and $G(.)$

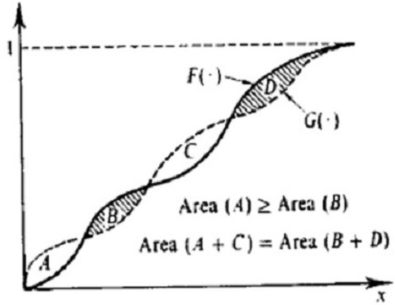


Figure: Probability distribution functions for lotteries $F(.)$ and $G(.)$

Second Order Stochastic Dominance

Definition

Second order stochastic dominance For any two lotteries (distributions) $F(.)$ and $G(.)$ with the same mean,$F(.)$ second order stochastically dominates lottery(or less risky than) $G(.)$ if, for every nondecreasing function $u : \mathbf{R} \rightarrow \mathbf{R}$, we have

$$\int u(x)dF(x) \geq \int u(x)dG(x)$$

.

State dependent utility function

- We begin by discussing a convenient framework for modeling uncertain alternatives that, in contrast to the lottery apparatus, recognizes underlying states of nature.
- State of Nature representation of Uncertainty
  - we show a state by $s \in S$ and its corresponding probability by $\pi_s > 0$
  - where $\sum_s \pi_s$
- Every uncertain alternative (which usually is a monetary return) is realized with a probability

Definition

Random variable: A random variable is a function $g : S \longrightarrow \mathbb{R}_+$ that maps states into monetary outcomes

- Contingent commodity, if state s occurs, then you will receive 1 \$.
- Example: If a bookmaker offers you odds of 10 to 1 against a certain horse winning, he is saying he will give you 10 if it wins and you will pay him 1 if it loses.

Definition

Extended expected utility representation: the preference relation $\succsim$ has an *extended expected utility representation* if for every $s \in S$, there is a function $u_s : \mathbb{R}^1_+ \longrightarrow \mathbb{R}$ such that for any $(x_1, ..., x_S) \in \mathbb{R}^S_+$ and $(x'_1, ..., x'_S) \in \mathbb{R}^S_+$,

$$(x_1, ..., x_S) \succsim (x'_1, ..., x'_S) \text{ if and only if } \sum_s \pi_s u_s(x_s) \geq \sum_s \pi_s u_s(x'_s). \blacksquare$$

State dependent utility function

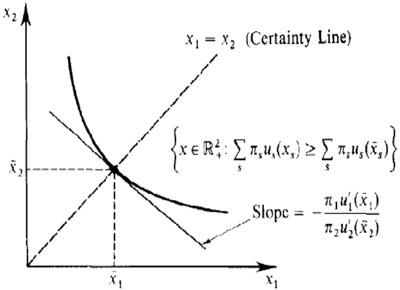


Figure: state dependent preferences

- The marginal rate of substitution at a point $(\bar{x}, \bar{x})$ is $\pi_1 u_1'(\bar{x})/u_2'(\bar{x})$.

State dependent utility function: Demand for insurance

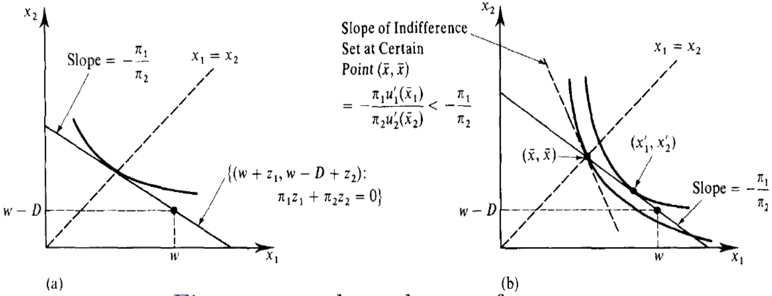


Figure: state dependent preferences

- The marginal rate of substitution at a point $(\bar{x}, \bar{x})$ for a state-dependent utility with non-uniform utility in each state is $\pi_1 u_1'(\bar{x})/\pi_2 u_2'(\bar{x}) < \pi_1/\pi_2$.