# Microeconomics II
# Chapter Two: Auction Theory, Part one

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# Outline

Introduction

Auctions as allocation mechanisms

Equilibrium bidding behavior in First-price auctions

Second-price auction

Effciency in auctions

Common-value auctions

# introduction

- This part offers an introduction to auctions, emphasizing:
  - Optimal bidding behavior in the first and second price options
  - Bidding strategies in common value auctions
  - Winner's curse
- This part only assumes a basic knowledge of algebra and calculus, and uses worked-out examples and figures
- Auctions have always been a large part of the economic landscape
  - Babylon in about 500 B.C.
  - Roman Empire, in 193 A.D.
  - Auctions houses like Sotheby's and Christie's were founded as early as 1744 and 1766, respectively

# Auction over the History

- The Babylonian Marriage Market: An Auction of Women in the Ancient World

Figure: In the 5th century BC, Greek Historian Herodotus wrote about the customs and traditions he witnessed while in Babylon. One of the more controversial customs he reports on is the Babylonian marriage market in which young women were gathered up and an ?auctioneer would get each of the women to stand up one by one, and he would put her up for sale.

# Roman Empire, in 193 A.D.

- Bodyguard of the Roman emperor, after killing Pertinax, the emperor, announced that the highest bidder could claim the Empire.

Figure: Pertinax

- Didius Julianus was the winner, becoming the emperor for two short months, after which he was beheaded.

# Roman Empire, in 193 A.D., Cont'

- Julianus was killed in the palace by a soldier in the third month of his holding royal office (1 June 193)

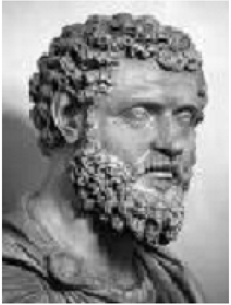

Figure: Didius Julianus

# Auctions Houses

- Sotheby's, Christie's (were founded as early as 1744 and 1766, respectively) and Other Auction Houses adapt to serve the next generation.
- In the late 1990's Sotheby's and Christies also experimented doing online auctions.

Figure: A book sale in progress at Messrs Sotheby, Wilkinson and Hodge of Wellington Street, 1888.

# Auctions Houses

- Sotheby's is a British multinational corporation headquartered in New York City. One of the world's largest brokers of fine and decorative art, jewelry, real estate, and collectibles, Sotheby's operation is divided into three segments: auction, finance, and dealer. The company's services range from corporate art services to private sales.

- Sotheby's decision to relocate its North American headquarters from Madison Avenue to a former cigar factory at 1334 York Avenue, New York, in 1982. The auction house closed its Madison Avenue galleries at East 76th Street. The Los Angeles galleries were sold and auctions of West Coast material moved to New York.

# Sotheby's New Location

Figure: York Avenue headquarters, New York City.

# ebay

- Commonly used auctions nowadays, are often online, with popular websites such as eBay with:
  - 11 billion US $ in total revenue
  - more than 27,000 employees worldwide

# Auctions by Governments

- Auctions have also been used by governments throughout history
- Auctioning bonds
- In the last decade Governments started to sell air waves 3G technology by auction
- The British 3G Telecom licenses generated Euro 36 billion in what British economists called the biggest auction ever

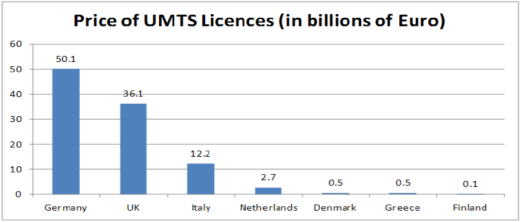


Figure: Prices of 3G licences.

# General setting

- Consider $N$ bidders who seek to acquire a certain object
- Each bidder $i$ has a valuation $v_i$ for the object
- Assume that there is one seller
  - First-price auction (FPA), whereby the winner is the bidder submitting the highest bid, and he/she must pay the highest bid (which in this case is his/hers).
  - Second-price auction (SPA), where the winner is the bidder submitting the highest bid, but in this case he/she must pay the second highest bid.
  - Third-price auction, where the winner is still the bidder submitting the highest bid, but now he/she must pay the third highest bid.

# Common Features of the Auctions

- *An allocation rule:* Specifying who gets the object.
  - The allocation rule for most auctions determines that the object is allocated to the bidder submitting the highest bid.
- *A payment rule:* How much every bidder must pay?
  - The payment rule in the FPA determines that the individual submitting the highest bid pays his own bid, while everybody else pays zero.
  - In contrast, the payment rule in the SPA specifies that the individual submitting the highest bid (the winner) pays the second-highest bid, while everybody else pays zero.

# Privately observed valuations

- Every bidder knows his/her own valuation for the object, $v_i$, but does not observe other bidder $j$'s valuation, $j \neq i$.

- Despite not observing $j'$s valuation, bidder $i$ knows the probability distribution behind bidder $j'$s valuation.
  - For instance, $v_j$ can be relatively high, e.g., $v_j = 10$, with probability 0.4, or low, $v_j = 5$, otherwise (with probability 0.6).

- More generally, bidder $j'$s valuation, $v_j$, is distributed according to a cumulative distribution function $F(v) = prob(v_j < v)$, intuitively representing that the probability that $v_j$ lies below a certain cutoff $v$ is exactly $F(v)$.

# Privately observed valuations, cont.

- For simplicity, it is assumed that every bidder's valuation for the object is drawn from a uniform distribution function between 0 and 1, i.e.,$v_j \sim U[0;1]$.
  - Where, probability of $v_j < v$, is $prob(v_j < v) = F(v)$. In the case of a uniform distribution requires $F(v) = v$.
  - Similarly, the valuations to the right-hand side of $v$ describe points where $v_j > v$ and, thus, bidder $j'$ valuation is higher than that of bidder $i$.
  - Mapping these points into the vertical axis we obtain the probability $prob(v_j > v) = 1 - F(v)$ which, under a uniform distribution, implies $1 - F(v) = 1 - v$.

# Graphical Illustration

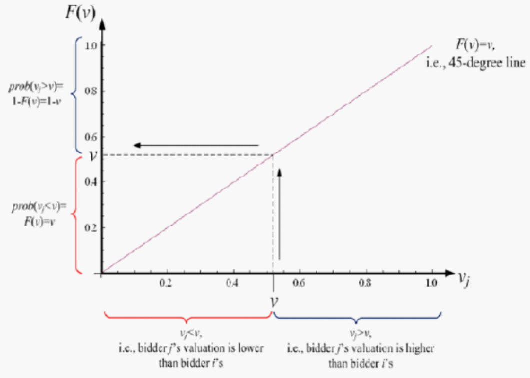


Figure: Uniform probability distribution.

# All bidders are ex-ante symmetric

- All bidders are using the same bidding function, $b_i : [0;1] \to \mathbb{R}_+$, for instance $b_i(v_i) = v_i/2$.
- This does not mean that all of them submit the same bid. Different valuation gives different bid.
- Consider $v_1 = 0.4$ and $v_2 = 0.9$ which give $b_1 = 0.2$ and $b_2 = 0.45$
- Bidders are symmetric in the bidding function they use, they can be asymmetric in the actual bid they submit.

# All bidders are ex-ante symmetric

- First, submitting a bid above one's valuation, $b_i > v_i$, is a **dominated** strategy.

$$Eu_i(b_i \mid v_i) = prob(win).(v_i - b_i) + prob(lose).0$$

- Because regardless of the probability of winning, the expected utility is negative.
- Similarly, submitting a bid $b_i$ that exactly coinsids with one'e valuation, $b_i = v_i$, also is **dominated** strategy, because with a deviation from this sterategy he loses nothing.
- Therefore the Equilibrium bidding strategy in a FPA must be $b_i < v_i$

# Bid shading in the FPA

- Bidder $i'$s valuation is $v_i$, his bid must be a fraction of his true valuation, $b_i = a.v_i$, where $a \in (0, 1)$.

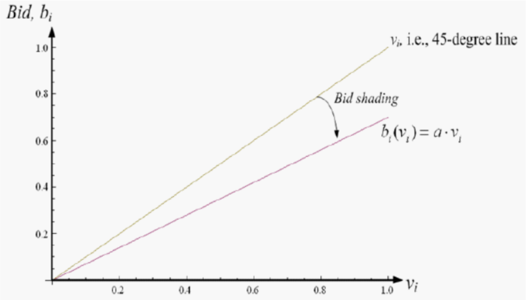


Figure: Bid shading in the FPA.

# Optimal Bidding sterategy in the FPA for Risk Neutral Bidder

- bidder $i'$s expected utility from submitting a given bid $x$, when his valuation for the object is $v_i$, is defined as:

$$Eu_i(b_i \mid v_i) = prob(win).(v_i - x) + prob(lose).0$$

- Probability of winning, $prob(win)$, and $x = b_i(v_i) = a.v_i$
- Solving for $v_i$ in $x = a.v_i$ yields $v_i = x/a$.
- Hence, the probability of winning is given by $prob(b_i > b_j)$ and, $prob(b_i > b_j) = prob(x > b_j)$.
- We obtain that $prob(b_i > b_j) = prob(x/a > v_j)$.
- We can now plug this probability of winning into bidder $i'$s expected utility from submitting a bid of $x$ in the FPA, as follows

$$Eu_i(x \mid v_i) = \frac{x}{a}.(v_i - x) = \frac{v_i x - x^2}{a}$$

# Bid shading intensity in the FPA

- Then, the optimal value from F.O.C of expected utility maximization results as:

$$x_i(v_i) = v_i/2$$

- Recall the definition of **pure strategy Bayesian Nash Equilibrium**

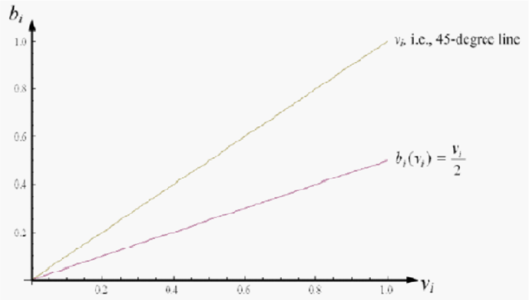


Figure: Optimal bidding function with $N = 2$ bidders.

# Extending the first-price auction to $N$ bidders

- Assuming that $v_1 < v_2 < \cdots < v_{N-1} < v_N$, probability of bidder $i$ winning when submitting a bid of \$$x$ is

  $$prob(win) = prob(\frac{x}{a} > v_1). \cdots .prob(\frac{x}{a} > v_n). \cdots .prob(\frac{x}{a} > v_{N-1})) = \frac{x}{a}. \cdots .\frac{x}{a}. \cdots .\frac{x}{a} = (\frac{x}{a})^{N-1}.$$

- Bidder $i'$s expected utility from submitting $x$ becomes

  $$E(u_i(x \mid v_i)) = (\frac{x}{a})^{N-1}(v_i - x) + [(1 - \frac{x}{a})^{N-1}].0$$

- The Optimal Value when more bidders participate in the auction is $x(v_i) = \frac{N-1}{N} v_i$

# Bid shading intensity in the FPA

- When more bidders participate in the auction, bidding functions approach the 45-degree line.

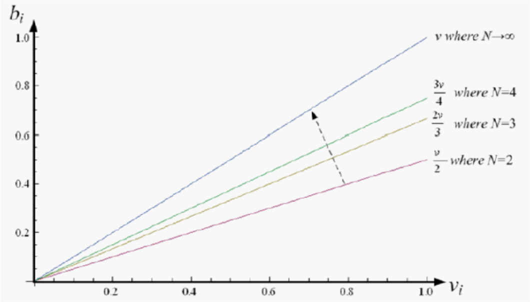


Figure: Optimal bidding function increases in $N$. Competition gets tougher as more bidders participate and, as a consequence, every bidder must increase his bid, ultimately ameliorating his incentives to practice bid shading.

# First-price auctions with risk-averse bidders

- How our equilibrium results would be affected if bidders are risk averse?
- The Bernoulli utility function is assumed as $u(z) = z^\alpha$ with risk aversion parametre $\alpha \in (0, 1]$
- For $N = 2$ and bid function $b(v_i) = a.v_i$ one can get:
$$E(u_i(x \mid v_i)) = \tfrac{x}{a}.(v_i - x)^\alpha$$
- Fisrt order condition with respect to the bid $x$ gives the optimal level of biding $x(v_i) = \frac{v_i}{1+\alpha}$
- Intuitively, a risk-averse bidder submits more aggressive bids than a risk-neutral bidder in order to minimize the probability of losing the auction.

# Second-price auction: Equilibrium Outcome

- In the second-price auction, bidding your own valuation, i.e., $b_i(v_i) = v_i$, is a **weakly dominant** strategy for all players.
  - i.e., it yields a larger (or the same) payoff than submitting any other bid.
- submitting a bid $b_i(v_i) = v_i$ yields expected profit equal or above that from submitting any other bid, $b_i(v_i) \neq v_i$.
- To show this bidding strategy is an equilibrium outcome of the SPA three cases are examined.
  - $b_i(v_i) = v_i$
  - $b_i(v_i) < v_i$
  - $b_i(v_i) > v_i$

# Second-price auction: Equilibrium Outcome

- Regardless of the valuation you assign to the object, and independently on your opponents' valuations, submitting a bid $b_i(v_i) = v_i$ yields expected profit equal or above that from submitting any other bid, $b_i(v_i) \neq v_i$.

- We can then compare which bidding strategy yields the largest expected payoff

# Second-price auction: Equilibrium Outcome

- *Case one:* If the bidder submits his own valuation, $b_i(v_i) = v_i$, then three situations can arise.
- $h_i = max_{j \neq i}\{b_j\}$ is the highest bid among all bidders different from bidder $i$, $j \neq i$.

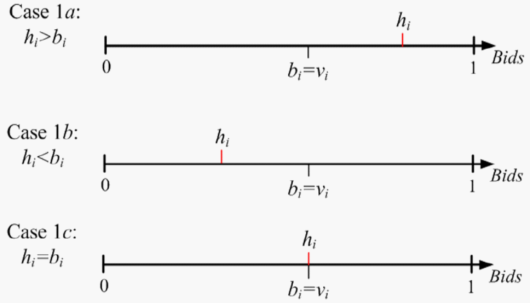


Figure: Cases arising when $b_i(v_i) = v_i$.

# Second-price auction: Equilibrium Outcome

- If his bid lies below the highest competing bid, i.e., $b_i < h_i$, where $h_i = max_{j \neq i}\{b_j\}$.
  - then bidder $i$ loses the auction, obtaining a zero payoff
- If his bid lies above the highest competing bid, i.e., $b_i > h_i$, then bidder $i$ wins, paying a price of $h_i$
  - He obtains a net payoff of $v_i - h_i$
- If, instead, his bid coincides with the highest competing bid, i.e., $b_i = h_i$, then a tie occurs, and
  - the object is randomly assigned, yielding an expected payoff of $\frac{1}{2}(v_i - h_i)$ for player $i$.

# Second-price auction: Deviation from Equilibrium

- *Case two:* bidder $i$ obtains the same payoff submitting a bid that coincides with his privately observed valuation for the object ($b_i(v_i) = v_i$, as in the First case) and shading his bid ($b_i(v_i) < v_i$). Therefore, he does not have incentives to conceal his bid, since his payoff would not improve from doing so.

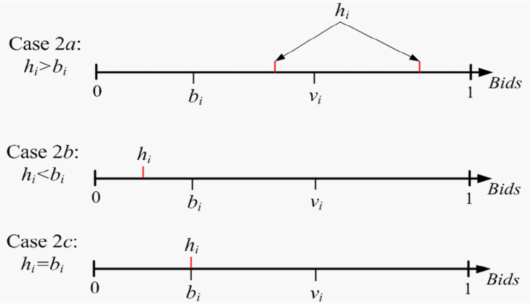


Figure: Cases arising when $b_i(v_i) < v_i$.

# Second-price auction: Deviation from Equilibrium

- If his bid lies below the highest competing bid, i.e., $b_i(v_i) < v_i$, then bidder $i$ loses the auction, obtaining a zero payoff.
- If his bid lies above the highest competing bid, i.e., $b_i(v_i) > v_i$, then bidder $i$ wins the auction, obtaining a net payoff of $v_i?h_i$.
- If, instead, his bid coincides with the highest competing bid, i.e., $b_i(v_i) = v_i$, then a tie occurs, and the object is randomly assigned, yielding an expected payoff of $\frac{1}{2}(v_i - h_i)$.

# Second-price auction: Deviation from Equilibrium

- *Case three:* Hence, bidder $i'$s payoff from submitting a bid above his valuation either coincides with his payoff from submitting his own value for the object, or becomes strictly lower, thus nullifying his incentives to deviate from his equilibrium bid of $b_i(v_i) = v_i$.

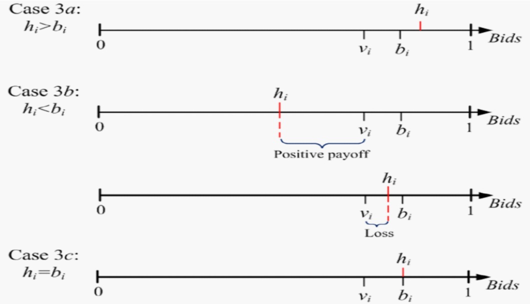


Figure: Cases arising when $b_i(v_i) > v_i$.

# Second-price auction: Deviation from Equilibrium

- If his bid lies below the highest competing bid, i.e., $b_i(v_i) < v_i$, then bidder $i$ loses the auction, obtaining a zero payoff.
- If his bid lies above the highest competing bid, i.e., $b_i(v_i) > v_i$, then bidder $i$ wins the auction. In this scenario, his payoff becomes $v_i - h_i$, which is positive if $v_i > h_i$, or negative otherwise.
  - The latter case, in which bidder $i$ wins the auction but at a loss (negative expected payoff), did not exist in our above analysis of $b_i(v_i) = v_i$ and $b_i(v_i) < v_i$, since players did not submit bids above their own valuation.
  - Intuitively, the possibility of a negative payoff arises because bidder i's valuation can lie below the second highest bid, as figure 11 illustrates, where $v_i < h_i < b_i(v_i)$.

# Second-price auction: Deviation from Equilibrium

- If, instead, his bid coincides with the highest competing bid, i.e., $b_i(v_i) = h_i$, then a tie occurs, and the object is randomly assigned, yielding an expected payoff of $\frac{1}{2}(v_i - h_i)$. Similarly as our above discussion, this expected payoff is positive if $v_i > h_i$, but negative otherwise.

# Effciency

## Definition

Auctions, and generally allocation mechanisms, are characterized as efficient if the bidder (or agent) with the highest valuation for the object is indeed the person receiving the object.

- Intuitively, if this property does not hold, the outcome of the auction (i.e., the allocation of the object) would open the door to negotiations and arbitrage among the winning bidder who, despite obtaining the object, is not the player who assigns the highest value to it and other bidder(s) with higher valuations who would like to buy the object from him.

# Efficiency in auctions

- According to this criterion, both the FPA and the SPA are efficient, since the bidder with the highest valuation submits the highest bid, and the object is ultimately assigned to the player who submits the highest bid.

# Common-value auctions

- The auction formats considered above assume that each bidders privately observes his own valuation for the object.

- The auction formats considered implying that two bidders are unlikely to assign the same value to the object for sale.

- In some auctions, such as the **government sale of oil leases, 3G Spectrum**, bidders (oil companies) might assign the same monetary value to the object (**common value**).
  - i.e., the profits they would obtain from exploiting the oil reservoir.

- Bidders are unable to precisely observe the value of this oil reservoir but, instead, gather estimates of its value.

# Oil Fields Lease

## Example

- Firms cannot accurately observe the exact volume of oil in the reservoir, or how difficult it will be to extract.
- They can accumulate different estimates from their own engineers, or from other consulting companies, that inform the firm about the potential profits to be made from the oil lease.
- Imprecise estimates of profits, $v$, $v \in \{10, 11, \cdots, 20\}$ in millions of dollars.
- Oil company **A** hires a consultant, and gets a signal (a report), $s$, as follows

$$s = \begin{cases} v + 2, & \text{with prob, 0.5, and} \\ v - 2, & \text{with prob 0.5} \end{cases} \tag{1}$$

# Oil Fields Lease

## Example

- Alternatively the conditional probability that the true value of the oil lease is $v$, given that the firm receives a signal $s$, is

$$prob(v \mid s) = \begin{cases} \frac{1}{2}, & \text{if} v = s - 2 (\text{overestimate}), \text{ and} \\ \frac{1}{2}, & \text{if} v = s + 2 (\text{underestimate}) \end{cases} \tag{2}$$

- The true value of the lease is overestimated when $v = s - 2$, i.e., $s = v + 2$ and the signal is above $v$;
- and it is underestimated when $v = s + 2$, i.e., $s = v - 2$ and the signal lies below $v$.

# Oil Fields Lease

### Example

- If company A was not participating in the auction, then the expected value of the oil lease would be $s = 0.5(s - 2) + 0.5(s + 2)$.
- implying that the firm would pay for the oil lease a price $p < s$, making a positive expected profit.