Microeconomics I for Ph.D.
Chapter Four: Equilibrium and Time

GholamReza Keshavarz-Haddad

Sharif University of Technology, GSME
G.K.Haddad@sharif.edu

December 16, 2019

Outline

Introduction

- This chapter presents the basic elements of the extension of competetive equilibrium theory to an inter-temporal setting.

Inter-temporal Utility

- **Assumptions:**
  - There are infinitely many dates $t = 0, 1, ...,$
  - The objects of choice for consumers are consumption streams $c = (c_0, ..., c_t, ...)$, where $c_t \in \mathbb{R}_L^+, c_t \geqq 0$.
  - The consumption streams are bounded, that is, $\| c_t \| < \infty$.
  - Preferences over consumption streams $c = (c_0, ..., c_t, ...)$ can be represented by a utility function $V(c)$ as follows:

$$V(c) = \sum_{t=0}^{\infty} \delta^t u(c_t) \tag{1}$$

  - Where $\delta < 1$ is a discount factor, $\delta = 1/(1 + r)$, and $u(.)$ which is defined on $\mathbb{R}_+^L$, is strictly increasing and concave.

Inter-temporal Utility

- **Some Notations and Concepts:**
  - Given a consumption stream $c = (c_0, ..., c_t, ...)$, we let $c^T = (c_0^T, c_1^T, ...)$ denote the T-period "*back-ward shift*" consumption stream, Namely $c_t^T = c_{t+T}$ for all $t \geq 0$, **in date t she consumes c_{t+T}.**

- **(1)Time impatience**, is the relative valuation placed on a good at an earlier date compared with its valuation at a later date.

- The requirement that future utility is discounted, implies time impatience.

- If $c = (c_0, ..., c_t, ...)$ then the "*forward shifted*" consumption stream $c' = (0, c_0, ..., c_{t-1}, ...)$ is strictly worse than c, **in date t she consumes c_{t-1}.**

Intertemporal Utility

Proof.

- Let c be a nonzero consumption stream, then by concavity of $u(.)$ we have, $u(c_t) > u(0)$ for some t, because $u(.)$ is strictly increasing.

$$V(c) = \sum_{t=0}^{\infty} \delta^t u(c_t) > \sum_{t=0}^{\infty} \delta^t u(0) = (1-\delta)^{-1} u(0) \qquad (2)$$

- Therefore $u(0) + \delta V(c) < V(c)$.
- If $c' = (0, c_0, ..., c_{t-1}, ...)$, then;

$$\begin{aligned} V(c') &= u(0) + \sum_{t=1}^{\infty} \delta^t u(c_{t-1}) \\ &= u(0) + \delta \sum_{t=0}^{\infty} \delta^t u(c_t) = u(0) + \delta V(c) \end{aligned} \qquad (3)$$

Concepts, Cont.:

- Hence $V(c') < V(c)$. Hence $V(.)$ exhibits *time impatience.*
- So the utility function allows us to compare any two consumption streams.
- The δ can be interpreted as the probability of survival to the next period. Then $V(c)$ is the expected value of life time utility.

Stationarity

- **(2) Stationarity:** A more general form of the utility function would be;

$$V(c) = \sum_{t=0}^{\infty} u_t(c_t) \tag{4}$$

- The form 1 is the special case of 4 in which $u_t(c_t) = \delta^t u(c_t)$.
- What is the Stationarity?
- Consider two consumption streams $c \neq c'$ such that $c_t = c'_t$ for $t \leq T - 1$, but the consumption streams differ from T.
- The problem of choosing at $t = T$ between the current and future consumptions in c and c' is the same problem that a consumer would face at $t = 0$ in choosing between the streams c^T and c'^T.
- The c^T and c'^T are the T backward shifts of c and c'.
- Then the Stationarity requires that:

$$V(c) \geq V(c') \text{ if only if } V(c^T) \geq V(c'_T)$$

Proof.

- Let c and c' be two consumption streams such that $c^T = c'^T$ for every $t \leq T-1$. Then

$$\begin{aligned} V(c) - V(c') &= \sum_{t=0}^{\infty} \delta^t u(c_t) - \sum_{t=0}^{\infty} \delta^t u(c'_t) \\ &= \sum_{t=T}^{\infty} \delta^t u(c_t) - \sum_{t=T}^{\infty} \delta^t u(c'_t) \\ &= \delta^T [\sum_{t=0}^{\infty} \delta^t u(c_{t+T}) - \sum_{t=0}^{\infty} \delta^t u(c'_{t+T})] \\ &= \delta^T [V(c^T) - V(c'^T)] \end{aligned} \tag{5}$$

- Hence $V(c) - V(c') \geq 0$ if only if $V(c^T) - V(c'^T) \geq 0$. So stationarity holds. □

Additive Separability

- **(3)Additive Separability.** Suppose $F(.)$ is a function of $c_0, ..., c_T$ variables. We say that$F(.)$ is completely additively separable if there exist functions $F_1, ..., F_T$, each a function of one variable, such that:
  $F(c_0, ..., c_T) = F_0(c_0) + F_1(c_1) + ... + F_T(c_T)$.

- Two implications of the additive form of utility function at any time date $T$are:
  1. The induced ordering on consumption streams at date $T + 1$ is independent of the consumption stream followed from 0 to T.
  2. The ordering on consumption streams from 0 to T is independent of weather consumption expectation we may have from $T + 1$ onward.

- If the preference ordering over consumption streams satisfies these two properties then one can represent the preferences by a utility function of the form $V(c) = \sum_t u_t(c_t)$.

Length of period

- How plausible is the separability assumption?
- It depends on the length of the period.
- What determines length of the period?
- **(4) Length of period.** It should be interval of time for which prices can be taken as constant.

Recursive utility

- **Recursive utility.** From the 1 for the utility function, we have $V(c) = u(c_0) + \delta V(c^1)$ for any consumption stream $c = (c_0, c_1, ..., c_t, ...)$.

- If we think of $u = u(c_0)$ as current utility and of $V = V(c^1)$ as future utility, then the marginal Rate of Substitution of current for future utility equals δ and is therefore independent of the levels of current and future utility.

- δ is independent of the utility levels.

- In a more general representation we may have: $G(u, V) = u + \delta V$. This utility function has the property that the ordering of future consumption streams is independent of the consumption stream followed in the past.

Altruism

- **Altruism.** The $\delta < 1$ means that the members of the current generation care for their children, but not quite as much as for themselves.

- If generation lives a single period and we think of generation 0 as enjoying her consumption according to $u(c_0)$, but caring about the *utility* $V(c^1)$ of the nest generation according $\delta V(c^1)$, then

$$V(c) = u(c_0) + \delta V(c^1)$$

is her overall utility function.

Inter-temporal Production and Efficiency

- **Assumptions:**
  1. There is infinite sequence of dates $t = 0, 1, ...$
  2. In each period there are L commodities, for instance $L = 2$ labor services and a generalized consumption-investment good.
  3. the goods are *non-durables*.
  4. there is a production technology that uses the endowed labor and consumption-investment goods to produce products, recall the production set from chapter 5.
  5. the technological possibilities at t will be formally specified by a production set $Y \subset \mathbb{R}^{2L}$ whose production plans is written $y = (y_b, y_a)$.
  6. Where b and a stands for *before* and *after* respectively, with $y_b \in \mathbb{R}^L$ and $y_a \in \mathbb{R}^L$.

- Furthermore Some assumptions are imposed on Y.
    1. Y is closed and convex
    2. No free lunch $Y \cap \mathbb{R}_+^{2L} = \{0\}$
    3. Free disposal $Y - \mathbb{R}_+^{2L} \subset Y$
    4. Production takes time. If $y = (y_b, y_a) \in Y$ then $(y_b, 0)$, *possibility of truncation.*

No Free lunch

- No free lunch $Y \cap \mathbb{R}^{2L}_{+} = \{0\}$

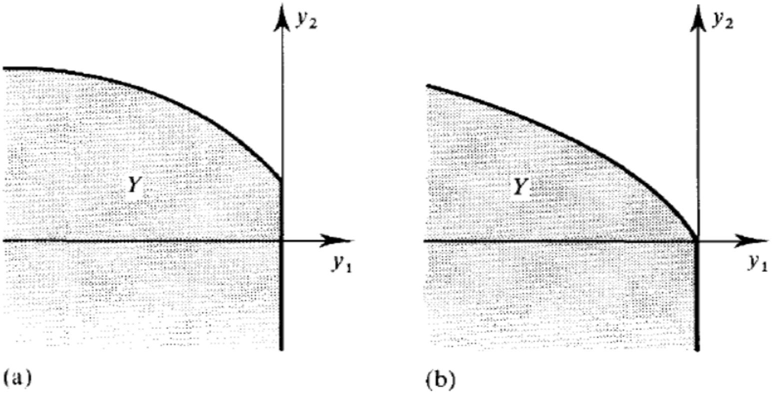

Figure: *a*. Violates no-free lunch, *b*. satisfies no-free lunch

Free disposal

- Free disposal $Y - \mathbb{R}^{2L}_{+} \subset Y$. This means that the firm can always throw away inputs if it wants.

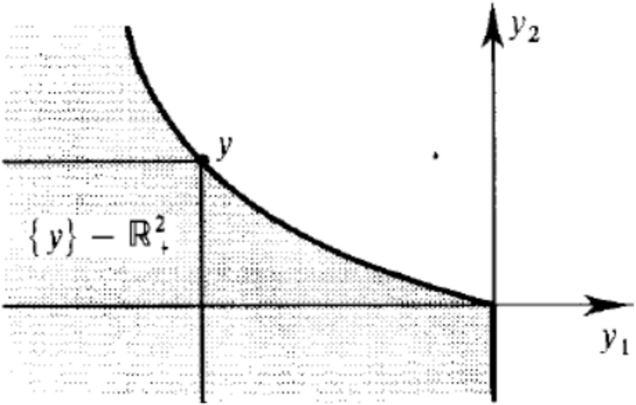

Figure: Free disposal for $L = 2$

Free disposal

- The meaning of this is that for any point in Y , points that use less of all components are also in Y.
- Thus if $y \in Y$, any point below and to the left is also in Y. The conclusion is that the production set is unbounded as you move down and to the left.

Ramsey-Solow Model

- There are two commodities, labour and consumption-investment good.
- in period t_0 the two inputs are applied and in period t_1 output x is available.
- The production technology is defined as $F(k, l)$.

$$Y = \{(-k, -l, x, 0) : k \geq 0, l \geq 0, x \leq F(k, l)\} - \mathbb{R}^4_+$$

- Labour is a primary factor;and it can not be produced.

Cost-of-Adjustment Model

- suppose that there three goods in te initial period:
  - capacity k
  - consumption good x
  - labour l
- consumption good is produced at the last period by $F(k, l)$.
- this output can be transfered into invested capacity at last period at a cost of $k' + \gamma(k' - k)$ units of consumption goodfor k' units of capacity, where;
  - $\gamma(.)$ is a convex function satisfying $\gamma(k' - k) = 0$ for $k' < k$ and $\gamma(k' - k) > 0$ for $k' > k$
  - the term $\gamma(k' - k)$ reperesents the cost of sdjusting capacity upward in a given period relative to the previous period.
- formally the production set Y is;$Y = \{(-k, 0, -l, k', x, 0) : k \geq 0, l \geq 0, k' \geq 0, x \leq F(k, l) - k' - \gamma(k' - k)\} - \mathbb{R}^6_+$

production path or production program

- Once our technolgy has been specified, we can define define what constitutes a path of production plans:

Definition

production path or production program. The list of $(y_0, y_1, ..., y_t, ...)$ is a production path or production program, or production trejectory, if $y_t \in Y \subset \mathbb{R}^{2L}$ for every t.

- Some points form production path:
  - there is overlap in the time indices over whicvh the production palns y_{t-1} and y_t.
  - Both $y_{a,t-1} \in \mathbb{R}^L$ and $y_{b,t}$ are **plans** which are made at dates $t-1$ and t respectively.
  - we have, at every t, a net input-output vector equal to

$$y_{a,t-1} + y_{b,t} \in \mathbb{R}^L$$

Efficient Production Path

- Some points form production path: Cont.
  - at $t = 0$, is assumed $y_{a,-1} = 0$
  - a negative entery in $y_{a,t-1}$ and $y_{b,t}$ is input and a positive one is output.
  - recall that $y_t = (y_{b,t}, y_{a,t})$, but we use $y_{a,t}$ as a part of input at date $t + 1$.

Definition

Efficient Production Path. The production path $(y_0, y_1, ..., y_t, ...)$ is efficient if there is no other production path $(y'_0, y'_1, ..., y'_t, ...)$ such that

$$y_{a,t-1} + y_{b,t} \leq y'_{a,t-1} + y'_{b,t} \text{ for all } t,$$

and equality does not hold for at least one t. Recall the definition of efficient production plan in chapter 5.

Profit level of a production path

Definition

Given a path $(y_0, y_1, ..., y_t, ...)$ and a price sequence $(p_0, p_1, ..., p_t, ...)$, the profit level associated with the production plan at t is

$$p_t . y_{b,t} + p_{t+1} . y_{a,t}.$$

- Recall from proposition 5-F-1 according which every profit maximizing production plan is efficient.
- we now follow the implications of profit maximization on the production plan made period by preiod.

Myopically profit maximizing production path and pareto optimality

Definition

Short Run profit maximizing Production Path. The production path $(y_0, y_1, ..., y_t, ...)$ is myopiclly, or short run, profit maximizing for the price sequence $(p_0, p_1, ..., p_t, ...)$ if for every t we have

$$p_t.y_{b,t} + p_{t+1}.y_{a,t} \geq p_t.y'_{b,t} + p_{t+1}.y'_{a,t} \text{ for all } y_t \in Y,$$

the price vector which is capale of sustaining a path $(y_0, y_1, ..., y_t, ...)$ as myopically profit-maximizing are often called ***Malinvaud*** prices for the path.

- can we generalize implication of proposition 5-F-1 to the $(y_0, y_1, ..., y_t, ...)$ which is myopiclly, or in short run, profit maximizing for the price sequence $(p_0, p_1, ..., p_t, ...)$?

Myopically profit maximizing production path and pareto optimality

- or in other word, does the first welfare theorem hold for myopic profit maximizing $(y_0, y_1, ..., y_t, ...)$?
- In a finite-horizen economy this proposition holds, but for a infinite horisen it need not.

Example

Capital Over-accumulation. Let $L = 1$ and

$$Y = \{(-k, k') : k \geq 0, k' \leq k\} \subset \mathbb{R}^2.$$

where $y_t = (-1, 1)$ for all t, that is we always carry forward one unit of output.

- - Then $y_{a,-1} + y_{b,0} = 0 + (-1)$ and $y_{a,t-1} + y_{b,t} = 1 + (-1) = 0$ for all $t > 0$.
  - Is this production path efficient? NO!
  - Just consider a production path like $y'_t = (0, 0)$ for all t, which has $y'_{a,t-1} + y'_{b,t} = 0$ for all $t \geq 0$.
  - Because $y_{a,t-1} + y_{b,t} \leq y'_{a,t-1} + y'_{b,t}$ for each $t \geq 0$.
- How can we arrive in an efficient productin path in infinite horisen?
- efficiency obtains if the present value of the period t production plan for period $t + 1$ goes to zero, that is $p_{t+1}.y_{a,t} \longrightarrow 0$ as $t \longrightarrow 0$.
- this condition is called **transversality condition**.
- ***This condition is violated in the example above.***

Theorem

Suppose that the production path $(y_0, ..., y_t, ...)$ *is myopically profit maximizing with respect to the price sequence* $(p_0, ..., p_t, ...) >> 0$*. Suppose also that the production path and the price sequence satisfy the* ***transversality condition*** $p_{t+1}.y_{a,t} \longrightarrow 0$*, and the path* $(y_0, ..., y_t, ...)$ *is efficient.*

Proof.

- Suppose that the path $(y'_0, ..., y'_t, ...)$ is such that $y_{a,t-1} + y_{b,t} \leq y'_{a,t-1} + y'_{b,t}$ for all t, with equality not holding for at least one t.
- Then there is $\epsilon > 0$ such that if we take a T sufficiently large for some strict inequality to correspond to a date previous to T, we must have

$$\sum_{t=0}^{T} p_t.(y'_{a,t-1} + y'_{b,t}) > \sum_{t=0}^{T} p_t.(y_{a,t-1} + y_{b,t}) + \epsilon$$

Theorem

Proof.

- if T is very large then $p_{T+1}.y_{a,T}$ is very small (because of transversality condition), therefore

$$\sum_{t=0}^{T} p_t.(y'_{a,t-1} + y'_{b,t}) > p_{T+1}.y_{a,T} + \sum_{t=0}^{T} p_t.(y_{a,t-1} + y_{b,t})$$

- by rearranging the terms in both sides of the inequality and Given that $y_{a,-1} = y'_{a,-1} = 0$, and from the possibility of truncation $(y'_{b,T}, 0) \in Y$ one can infer that;

$$p_T.y'_{b,T} + \sum_{t=0}^{T-1}(p_{t+1}.y'_{a,t} + p_t y'_{b,t}) > \sum_{t=0}^{T}(p_{t+1}.y_{a,t} + p_t.y_{b,t})$$

□

Proof.

- then we must have either

$$p_{t+1} \cdot y'_{a,t} + p_t y'_{b,t} > p_{t+1} \cdot y_{a,t} + p_t \cdot y_{b,t}$$

  for some $t \leq T-1$, or

$$p_T \cdot y'_{b,T} > p_{T+1} \cdot y_{a,T} + p_T \cdot y_{b,T}$$

  for $t = T$.
- In either case we arive in a violation of the myopic profit maximization assumption.
- **Therefore no such a $(y'_0, ..., y'_t, ...)$ can exist.**

□

The second fundamental welfare theorem in an inter-temporal context

- Given an efficient production path $(y_0, ..., y_t, ...)$, can it be price supported?
  - the question can be decompose into related parts:
    1. Is there a system of Malinvaud prices $(p_0, ..., p_t, ...)$ for $(y_0, ..., y_t, ...)$, that is, a sequence $(p_0, ..., p_t, ...)$ with respect to which $(y_0, ..., y_t, ...)$ is myopically profit maximizing?
    2. If the answer to (1) is yes, can we conclude that the pair $(y_0, ..., y_t, ...)$, $(p_0, ..., p_t, ...)$ satisfy the transversality condition?

Equilibrium: The One-Consumer Case

- This section bring the consumption and production sides together.
  - Assumptions:
    - Short run production function $Y \subset \mathbb{R}^{2L}$
    - A utility function $u(.)$ is defined on $\mathbb{R}^L_+$
    - A discount factor $\delta < 1$
    - A sequence of initial endowments$(\omega_0, ..., \omega_t, ...), \omega_t \in \mathbb{R}^L_+$
    - Y satisfies hypotheses (1) to (4) 20.C and that u(.) is strictly concave, differentiable,a nd has strictly positive marginal utilities throughout its domain.
    - Prices $(p_0, ..., p_t, ...)$ are given with $p_t \in \mathbb{R}^L_+$

Equilibrium: The One-Consumer Case

- Given a production path $(y_0, ..., y_t, ...), y_t \in Y$, the induced stream of consumptions $(c_0, ..., c_t, ...)$ is given by:

$$c_t = y_{a,t-1} + y_{bt} + \omega_t$$

production.

- If $c_t \geq 0$ for every t, then we say that the production path $(y_0, ..., y_t, ...)$ is *feasible.*
- Given a production path $(y_0, ..., y_t, ...)$ and a price sequence $(p_0, ..., p_t, ...)$, the induced stream of profits $(\pi_0, ..., \pi_t, ...)$ is given by

$$\pi_t = p_t . y_{bt} + p_{t+1} . y_{at}$$

for every t.

Equilibrium: The One-Consumer Case

- For fixed T and $\sum_{t=0}^{T} p_t.c_t = \sum_{t=0}^{T} p_t(y_{a,t-1} + y_{at} + \omega_t)$ we get

$$\sum_{t \leq T} (\pi_t + p_t.\omega_t) - \sum_{t \leq T} p_t.c_t = p_{T+1}.y_{aT}$$

Definition

The production path $(y_0^*, ..., y_t^*, ...)$, $y_t^* \in Y$ and the price sequence $p = (p_0, ..., p_t, ...)$ constitute a **Walrasian** or competitive equilibrium if:

1. $c_t^* = y_{a,t-1}^* + y_{bt}^* + \omega_t \geq 0$ for all t.

Example

For the trivial storage model we have $y_t = (y_{tb}, y_{ta}) = (-k, k')$ and $y_{a,t-1} + y_{bt} = 1 + (-1) = 0$.

Example

$$\begin{bmatrix} c_{1t}^* \\ c_{2t}^* \\ c_{3t}^* \end{bmatrix} = \begin{bmatrix} y_{a,1t-1}^* \\ y_{a,2t-1}^* \\ y_{a,3t-1}^* \end{bmatrix} + \begin{bmatrix} y_{b,1t}^* \\ y_{b,2t}^* \\ y_{b,3t}^* \end{bmatrix} + \begin{bmatrix} \omega_{1t} \\ \omega_{2t} \\ \omega_{3t} \end{bmatrix}$$

$$\begin{bmatrix} c_{1t}^* \\ c_{2t}^* \\ c_{3t}^* \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \\ 5 \end{bmatrix} + \begin{bmatrix} 17 \\ -5 \\ -4 \end{bmatrix} + \begin{bmatrix} \omega_{1t} \\ \omega_{2t} \\ \omega_{3t} \end{bmatrix}$$

Definition

2. For every t,

$$\pi_t = p_t.y^*_{bt} + p_{t+1}.y^*_{at} \geq p_t.y + p_{t+1}.y$$

for all $y = (y_b, y_a) \in Y$.

3. The consumption sequence $(c^*_0, ..., c^*_t, ...) \geq 0$ solves the problem:

$$Max \sum_t \delta^t u(c_t)$$

$$s.t. \sum_t p_t.c_t \leq \sum_t \pi_t + \sum_t p_t.\omega_t$$

Definition, Interpretations

1. Condition (1) is the feasibility requirement.
2. Condition (2) is the short run o myopic, profit maximization condition.
3. In condition (3) the budget constraint, the right hand side stands for the one consumer's total Wealth, and left hand side shows value of her total consumption stream over time.
4. At the equilibrium of the optimization problem consumptions the budget constraint must hold with equality.

- Important consequence of (4): At the equilibrium the transversality condition is satisfied. This point is argued concretely in the following **Proposition**.

Theorem

Suppose that the production path $(y_0^*, ..., y_t^*, ...)$ *and the price sequence* $(p_0, ..., p_t, ...)$*, constitute a Walrasian equilibrium. Then the* ***transversality condition*** $p_{t+1}.y_{a,t} \longrightarrow 0$ *holds, and the path* $(y_0, ..., y_t, ...)$ *is efficient.*

Proof.

- Denote $c_t^* = y_{a,t-1}^* + y_{bt}^* + \omega_t$. By

  $$\sum_{t \leq T} (\pi_t + p_t.\omega_t) - \sum_{t \leq T} p_t.c_t = p_{T+1}.y_{aT}$$

  since each of the sums in the left-hand side converges to $w < \infty$ as $T \longrightarrow \infty$ it is concluded that the transversality condition holds, $p_{t+1}.y_{a,t} \longrightarrow 0$.

□

Myopic utility maximization

Definition

It is said that the consumption stream $(c_0, \ldots, c_t, \ldots)$ is myopically utility maximizing in the budget set determined by $(p_0, \ldots, p_t, \ldots)$ and $W < \infty$ if utility cannot be increased by a new consumption stream that merely transfers purchasing power between some two consecutive periods.

Example

Show that a consumption stream $(c_0, ..., c_t, ...) >> 0$ is short-run utility maximizing for $(p_0, ..., p_t, ...)$ and $W < \infty$ if only if it satisfies $\sum_t p_t.c_t = W$ and the collection of first order conditions:

$$Max_{c_t, c_{t+1}} u(c_t, c_{t+1}) = Max_{c_t, c_{t+1}} \{u(c_t) + \delta u(c_{t+1})\}$$

$$s.t. p_t c_t + p_{t+1} c_{t+1} \leq W$$

for every t there is $\lambda_t > 0$ such that $\lambda_t p_t = \nabla u(c_t)$ and $\lambda_t p_{t+1} = \nabla u(c_{t+1})$. And finally,

$$\lambda p_t = \delta^t \nabla u(c_t)$$

for all t.

Theorem

If the consumption stream $(c_0, ..., c_t, ...)$ *satisfies* $\sum_t p_t.c_t = W < \infty$ *and condition*

$$\lambda p_t = \delta^t \nabla u(c_t)$$

for all t*, then it is utility maximizing in the budget set determined by* $(p_0, ..., p_t, ...)$ *and* W*.*

- Since Walrasian equilibrium is myopically profit maximizing and satisfies the transversality condition, we know that it is production efficient. Can we claim that the full first welfare theorem holds?

YES WE CAN.

- The statement implies that the utility maximization problem under the technological and endowment constraints give rise to Pareto optimality in consumption.

The first and second welfare theorems

Theorem

Walrasian equilibrium path $(y_0^*, ..., y_t^*, ...)$ *solves the planing problem*

$$Max_{c_t, c_{t+1}} \{\sum_t \delta^t u(c_t)\}$$

s.t. $c_t = y_{a,t-1} + y_{bt} + \omega_t \geq 0$ *and* $y_t \in Y$ *for all* t*.*

Theorem

Suppose that the path $(y_0^*, ..., y_t^*, ...)$ *solves the planing problem and that it yields strictly positive consumption,* $c_{lt} = y_{la,t-1}^* + y_{lbt}^* + \omega_{lt} > \epsilon$ *. Then the path is a Walrasian equilibrium with respect to some price sequence* $(p_0, ..., p_t, ...)$*.*