Microeconomics I for Ph.D.
Chapter Two: Equilibrium and its Basic Properties

GholamReza Keshavarz-Haddad

Sharif University of Technology, GSME
G.K.Haddad@sharif.edu

December 16, 2019

Outline

What topics will this chapter cover? Well, lets find them out.

- Pareto optimal allocation, an allocation with the property in which it is impossible to make any consumer better-off without making some other consumer worse-off.
- Walrasian Equilibrium, an economy with private ownership.In that economy consumers' budget set is determined by her owned endowments and her profit share from firms.
- A generalization of Walarasian Equilibrium, a price equilibrium with transfers, in this economy redistribution of wealth (endowments) among the consumers are possible.
- The remaining parts of the chapter are devoted to exploring the relationship between Pareto optimality and the equilibrium concepts.

Notations and definitions

- Assume an economy with I price taker consumers, J price taker firms and L commodities. Every consumer is a potential user of the L goods and there are L commodities in the production plan of each firm.
- Each consumer $i = 1, \dots, I$ has preference relation $\succsim_i$ which is defined on the consumption set $X_i \subset \mathbb{R}^L_+$.
  - The preference relation is assumed to be **Rational**, i. e. complete and transitive
- The economy's endowments are given to the consumer by a vector $\bar{\omega} = (\bar{\omega}_1, \dots, \bar{\omega}_L)$, in which $\bar{\omega}_l, l = 1, \dots, L$ is a vector with I elements. A typical element of $\bar{\omega}_l$ is ω_{li}.

Notations and definitions

- The preference relations are denoted by $\{(\succsim_i, X_i)\}_{i=1}^I$.
- The Production technologies in the economy are given by $\{Y_j\}_{j=1}^J$.
- And finally the resources are shown by $\bar{\omega}$
  - Therefore the whole economy is summarized by $(\{(\succsim_i, X_i)\}_{i=1}^I, \{Y_j\}_{j=1}^J, \bar{\omega})$

Example

The edgeworth box pure exchange economy is a case in which $L = 2$, $I = 2$, $X_1 = X_2 = \mathbb{R}_+^2$, $J = 1$ and $Y_1 = -\mathbb{R}_+^2$.

Definition

An economy is a *pure exchange economy* if its only technological possibility is that of *free disposal*, that is $Y_j = -\mathbb{R}_+^L, j = 1, \ldots, J$. The free disposal technology -no production can take place, only destruction.

Notations and definitions

Definition

Allocation: An allocation $(x, y) = (x_1, \ldots, x_I, y_1, \ldots, y_J)$ is a specification of consumption vector $x_i \in X_i \subset \mathbb{R}_+^L$ for each consumer and production plan $y_j \in Y_j \subset \mathbb{R}^L$ for each firm.

Definition

A Feasible Allocation: An allocation
$(x, y) = (x_1, \ldots, x_I, y_1, \ldots, y_J)$ is a feasible allocation
$\Sigma_i x_{li} = \bar{\omega}_l + \Sigma_j y_{lj}$ for every l. That is if $\Sigma_i x_i = \bar{\omega} + \Sigma_j y_j$.
the set of feasible allocation is denoted by:
$A = \{(x, y) \in X_1 \times \cdots \times X_I \times Y_1 \cdots \times Y_J : \Sigma_i x_i = \bar{\omega} + \Sigma_j y_j\} \subset \mathbb{R}^{L(I+J)}$.

Vilferedo Pareto

A socially desirable outcome is that of ***Pareto optimal*** allocation. The concept is named after Vilfredo Pareto (1848—1923), Italian engineer and economist, who used the concept in his studies of economic efficiency and income distribution.

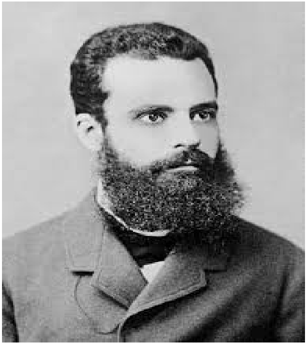


Figure 1: Vilferedo Pareto

Notations and definitions

Definition

A feasible allocation (x, y) is Pareto optimal (or Pareto efficient) if there is no other allocation $(x', y') \in A$ that Pareto dominates it, that is if there is no feasible allocation (x', y') such that $x' \succsim_i x_i$ for i and $x'_i \succ_i x_i$ for some i.

- In word, an allocation **is not Pareto optimal** if there is an alternative allocation where improvements can be made to at least one participant's well-being without reducing any other participant's well-being.

- If there is a transfer that satisfies this condition, the reallocation is called a **Pareto improvement**.

Definition

A feasible allocation x

- is called weakly Pareto efficient if there is no feasible allocation x' with $u_i(x_i') > u_i(x_i), (1 \leq i \leq I)$.
- is called strongly Pareto efficient if there is no feasible allocation x' with $u_i(x_i') \geq u_i(x_i), (1 \leq i \leq I)$ with at least one of these inequalities strict.

- A "weak Pareto optimum" (WPO) is an allocation for which there are no possible alternative allocations whose realization would cause every individual to gain.

- Thus, an alternative allocation is considered to be a Pareto improvement if and only if the alternative allocation is strictly preferred by all individuals.
- When contrasted with weak Pareto efficiency, a standard Pareto optimum as described above is referred to as a "strong Pareto optimum" (SPO).
- Weak Pareto-optimality is "weaker" than strong Pareto-optimality in the sense that any SPO also qualifies as a WPO, but a WPO allocation is not necessarily an SPO. **why?**

Proof.

Suppose that a feasible allocation (x, y) is strongly Pareto efficient, and take any allocation (x', y') for which $u_i(x') > u_i(x)$ for $\forall i$. Then, by SPO of (x, y), allocation (x', y') cannot be feasible.
Thus, (x, y) must also be weakly Pareto efficient. These are equivalent if $\succsim_i$ is strongly monotone. It suggests that (x, y) must be interior. $\square$

Definition

Constrained Pareto optimal. The condition of constrained Pareto optimality is a weaker version of the standard condition of Pareto optimality employed in economics, which accounts for the fact that a potential planner (e.g., the government) may not be able to improve upon a decentralized market outcome, even if that outcome is inefficient. This will occur if it is limited by the same informational or institutional constraints as are individual agents. Page 444 M.G.W

- An allocation is Pareto optimal if there is no waste:
    - That is, it is impossible to make any consumer strictly better off without making some other consumer worse off.
- Pareto optimality concept does not concern itself with **distributional** issues.
    - In a pure exchange economy, an allocation that gives all of society's endowments to one consumer who has **strongly monotone preferences is Pareto optimal**.

Notations and definitions

- Properties of competitive private ownership economies:
  - Every good is traded in a market at publicly know prices that consumers and firms take as unaffected by their own actions.
  - Consumers trade to maximize their well-bing,
  - Firms produce and trade to maximize their profits,
  - The wealth of consumers is derived from their ***endowments*** and from ownership claims (***shares***) to the profits of the firms.
  - i.e. Consumers are the owner of firms
- Consumer i owns commodities $\omega_i \in \mathbb{R}^L$ as her initial endowment vector.
- She claims to a share $\theta_{ij} \in [0,1]$ of the profit of firm j.
- Thus the Private Ownership Economy is summarized by $(\{(\succsim_i, X_i)\}_{i=1}^{I}, \{Y_j\}_{j=1}^{J}, \{\omega_i, \theta_{i1}, \ldots, \theta_{iJ}\}_{i=1}^{I})$.

Walrasian equilibrium

The price taking equilibrium for a private ownership economy is called ***Walrasian equilibrium***.

Definition

Walrasian equilibrium: Given a private ownership economy specified by $(\{(\succsim_i, X_i)\}_{i=1}^I, \{Y_j\}_{j=1}^J, \{\omega_i, \theta_{i1}, \dots, \theta_{iJ}\}_{i=1}^I)$, an allocation$(x^*, y^*)$ and a price vector $p = (p_1, \dots, p_L)$ constitute a *Walrasian* (or **competitive**) *equilibrium* if:

1. For every j, y_j^* maximizes profits in Y_j; that is,
$$p.y_j \leq p.y_j^* \text{ for all } y_j \in Y_j$$
2. For every i, x_i^* is maximal for $\succsim_i$ in the budget set
$$\{x_i \in X_i : p.x_i \leq p.\omega_i + \Sigma_j \theta_{ij} p.y_j^*\}.$$
3. $\Sigma_i x_i^* = \bar{\omega} + \Sigma_j y_j^*$.

Walrasian equilibrium, Interpretations

- Condition (1) of The definition (*Walrasian equilibrium*) says that at a Walrasian equilibrium, firms are maximizing their profits given the equilibrium price p, see chapter 5 MWG.
- Condition (2) implies that consumers are maximizing their well-being given, first, equilibrium prices and second, the wealth derived from endowments and their shares of profits, see chapter 3 MWG.
- Condition (3) says that markets must clear at an equilibrium price, i.e. **firms and consumers are subjected to the price vector**.

Price Equilibrium with Transfers

- The aim of this chapter is to relate of Pareto Optimality to supportability by means of price-taking behavior.
- We can imagine a situation where a social planner is able to carry out (lum-sum) redistributions of wealth in a desired manner.

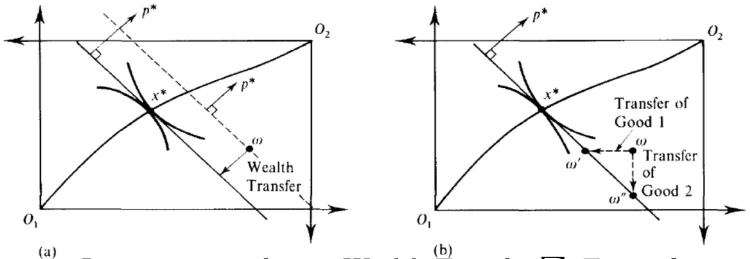


Figure 2: Lum-sum transfers **a**. Wealth Transfer $\sum_i T_i = 0$, **b**. Transfer of endowments

Price Equilibrium with Transfers

Definition

Walrasian equilibrium: Given a private ownership economy specified by $(\{(\succsim_i, X_i)\}_{i=1}^{I}, \{Y_j\}_{j=1}^{J}, \{\omega_i, \theta_{i1}, \ldots, \theta_{iJ}\}_{i=1}^{I})$, an allocation$(x^*, y^*)$ and a price vector $p = (p_1, \ldots, p_L) \gg \mathbf{0}$ constitute a *Price Equilibrium with Transfers* if there is an assignment of wealth levels $(w_1, \ldots, w_I)$ with $\Sigma_i w_i = p.\bar{\omega} + \Sigma_j p.y_j^*$. such that:

Price Equilibrium with Transfers

1. For every j, y_j^* maximizes profits in Y_j; that is,
$$p.y_j \leq p.y_j^* \text{ for all } y_j \in Y_j$$
2. For every i, x_i^* is maximal for $\succsim_i$ in the budget set
$$\{x_i \in X_i : p.x_i \leq w_i.$$
   - Where $w_i = p.\omega_i + \Sigma_j \theta_{ij} p.y_j^*$.

   ]
3. $\Sigma_i x_i^* = \bar{\omega} + \Sigma_j y_j^*$ for all $i = 1, \ldots, I$.

A Walrasian equilibrium is a special case of a price equilibrium with transfers.

Price Equilibrium with Transfers, Interpretations

- In a price equilibrium with transfers there must be wealth distribution such that allocation (x^*, y^*) and $p \in \mathbf{R}^{\mathbf{L}}_{++}$ constitute an equilibrium, i.e. is a **Walrasian equilibrium**.
- There is no supposition for consumers' wealth level.
- The **Walrasian equilibrium** is a special case of **Price Equilibrium with Transfers**, but without any transfers.
- Any desired Pareto optimal allocation can be achieved by appropriately redistributing wealth in a lump-sum fashion and then letting the market work (i.e. any Pareto optimal allocation is supportable as an equilibrium with transfers).

Example

Argue graphically that in an Edgeworth box economy with locally non-satiated preferences a Walrasian equilibrium is Pareto optimal.

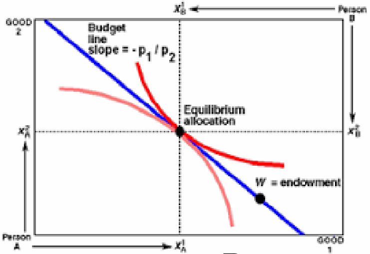


Figure 3: .B.3

Example

Consider an Edgeworth box economy in which the consumers have the CD utility functions $u_1(x_{11}, x_{21}) = x_{11}^{\alpha} x_{21}^{1-\alpha}$ and $u_2(x_{12}, x_{22}) = x_{12}^{\beta} x_{22}^{1-\beta}$. Consumer i's endowments are $(\omega_{1i}, \omega_{2i}) >> 0$, for $i = 1, 2$. Find the offer curves for the consumers, and equilibrium prices ratio.

Solution: The offer curve of consumer 1 can be derived as follows:

$$\max_{x_{11}, x_{21}} x_{11}^{\alpha} x_{21}^{1-\alpha}$$

s.t.

$$p_1 x_{11} + p_2 x_{21} = p_1 \omega_{11} + p_2 \omega_{21}$$

. From the F.O.C. it follows that $x_{11} = \frac{\alpha(p_1\omega_{11}+p_2\omega_{21})}{p_1}$, $x_{21} = \frac{(1-\alpha)(p_1\omega_{11}+p_2\omega_{21})}{p_2}$, $x_{12} = \frac{\beta(p_1\omega_{12}+p_2\omega_{22})}{p_1}$ and $x_{22} = \frac{(1-\beta)(p_1\omega_{12}+p_2\omega_{22})}{p_2}$. Go ahead and complete the solution!

Example

In a certain economy there are two commodities, education (e) and food (f), produced by using labour (L) and land (T) according to the production functions:

$$e = (min\{L, T\})^2 \text{ and } f = (LT)^{1/2}$$

There is a single consumer with the utility function

$$u(e, f) = e^{\alpha} f^{1-\alpha};$$

and endowment (ω_L, ω_T). To ease of the calculations, take $\omega_L = \omega_T = 1$ and $\alpha = 1/2$.

Find the optimal allocation of the endowments to their productive uses. Solution will be provided in the class.

The First Fundamental Theorem of Welfare Economics

- The theorem states **conditions under which** any price equilibrium with transfer (a **Walrasian Equilibrium**) is **Pareto optimum**.
- This specifies conditions under which a rational competitive equilibrium will be efficient.
- The theorem is the mathematical explanation for Adam Smith's **invisible hand**.
- A single, and very weak assumption, ***locally non-satiated preferences***, is all that is required for the result.

Definition

Locally non-satiated preference relation: The preference relation $\succsim_i$ on the consumption set X_i is ***locally non-satiated*** if for every $x_i \in X_i$ and every $\epsilon > 0$, there is an $x'_i \in X_i$ such that $\parallel x'_i - x_i \parallel \leq \epsilon$ and $x'_i \succ_i x_i$

Locally non-satiated preference relation

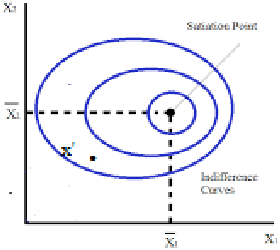


Figure 4: The property of local nonsatiation of consumer's preferences states that for any bundle of goods ***there is always*** another bundle of goods arbitrarily close to that and the bundle is preferred to it. What this means is that a consumer always either prefers more of an item or less of an item of goods.

Price Equilibrium with Transfers, Interpretations

- Local non-satiation is implied by monotonicity of preferences (a stronger condition). Because the converse isn't true, local non-satiation is a weaker condition.
- There is no requirement that the preferred bundle x_i' contain more of any good - hence, some goods can be "bads" and preferences can be non-monotone.
- if we have $x_i' \gg x_i$ (a stronger condition) this implies $x_i' \geq x$ and $x_i' \neq x$ (a weaker condition), coupled with strong monotonicity this implies $x_i' \succ x$ which is all we need to prove that strong monotonicity implies monotonicity (weaker).

The First Fundamental Theorem of Welfare Economics

Definition

If preferences are locally non-satiated (**LNS**), and if (x^*, y^*, p) is a price equilibrium with transfers, then the allocation (x^*, y^*) is Pareto optimal. In particular, any Walrasian equilibrium allocation is Pareto optimal.

Proof.

- Suppose that (x^*, y^*, p) is a price equilibrium with transfers and that the wealth levels are $(w_1, \ldots, w_I)$.
- Where $\Sigma_i w_i = p.\bar{\omega} + \Sigma_j p.y_j^*$.
- The preference maximization part of Price Equilibrium with Transfers implies that:

$$\text{If } x_i \succ_i x_i^* \text{ then } p.x_i > w_i \tag{3}$$

The First Fundamental Theorem of Welfare Economics

Proof.

- **LNS** together with the preference maximization implies that:

$$\text{If } x_i \succsim_i x_i^* \text{ then } p.x_i \geq w_i \tag{4}$$

  - This point can be proved by contradiction, i.e. $x_i \succsim_i x_i^*$ but $p.x_i < w_i$.
    - A predicate is:
    - Strong if there are few objects for which it is true; and
    - Weak if there are many objects for which it is true.
    - We say statement A is Stronger than B if A implies B.

1. Now, **by contradiction** consider an allocation that (x, y) that Pareto dominates (x^*, y^*).
2. Then by be definition (Pareto dominance), $x_i \succsim_i x_i^*$ for all i and $x_i \succ_i x_i^*$ for some i.

The First Fundamental Theorem of Welfare Economics

Proof.

Cont.

3. By (4), we must have $p.x_i \geq w_i$ for all i, and by (3) $p.x_i > w_i$ for some i.
4. Hence
$$\Sigma_i p.x_i > \sum_i w_i = p.\bar{\omega} + \Sigma_j p.y_j^*.$$
5. Since y_j^* is profit maximizing for firm j at prices p, we have $p.\bar{\omega} + \Sigma_j p.y_j^* \geq p.\bar{\omega} + \sum_j p.y_j$. Thus
$$\sum_i p.x_i > p.\bar{\omega} + \sum_j p.y_j.$$
6. But (x, y) cannot be a feasible allocation.

$\square$

The First Fundamental Theorem of Welfare Economics: Interpretations

- The importance of **LNS** assumption
- The Edgeworth box where locally nonsatiation fails for consumer 1 (the indifference is thick).

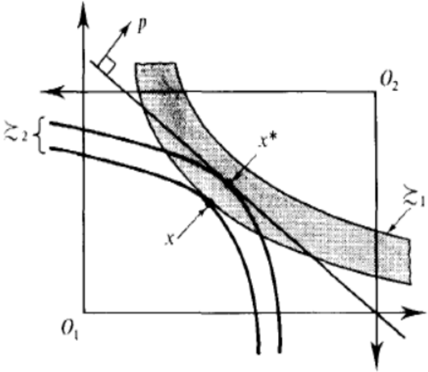


Figure 5: A price Equilibrium with transfers that is not a Pareto optimum, **Failure of the 1st welfare theorem.**

The First Fundamental Theorem of Welfare Economics: Interpretations

- In the above Edgworth box x^*, a price equilibrium for the price vector $p = (p_1, p_2)$ is not Pareto optimal, (why?)

- Consumer (1) is indifferent about a move to allocation x, and consumer (2), having strongly monotone preferences, is strictly better off, therefore the x^* cannot be Pareto optimal.

- Although assumption on the primitive of the economy are very weak i.e. only local non-satiation is required, we must keep in mind all the exogenous assumptions underlying the model:

The First Fundamental Theorem of Welfare Economics: Interpretations

1. Markets are complete
2. No externalities
3. No uncertainty
4. Price-taking behavior (very strong assumption when economy is small).

- In the next paragraph we will study the second fundamental theorem of welfare economics.
- The theorem gives conditions under which any desired distributional aims can be achieved through the use of competitive price taking markets.

The Second Fundamental Theorem of Welfare Economics

- The second Welfare Theorem is the more useful theorem, specifying conditions that are more strict under which a Pareto efficient allocation can be replicated by a competitive equilibrium.
- This is more useful because it is easier to solve for efficient allocations than competitive equilibria.
- It is important to stress, though, that the welfare theorems are just theorems. Their connection to the real world is limited.
- Adam Smith knew nothing about the theorems when he conceived the invisible hand metaphor. The theorems are just components of a model that helps to explain why invisible hand is true.

The Second Fundamental Theorem of Welfare Economics

- A two consumers Edgworth box economy in which preferences are convex. In the Equilibrium allocation x^* offer curves intersect

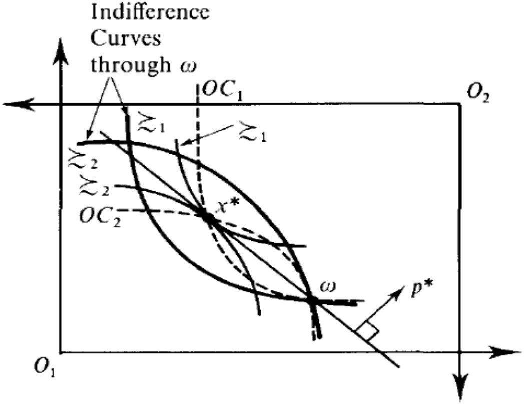


Figure 6: Walrasian Equilibrium with convex preferences.

The Second Fundamental Theorem of Welfare Economics

- A one consumer and one producer economy in which the Pareto optimality can not be supportable as an equilibrium $Max_{x_1,x_2} u(x_1, x_2), \quad s.t. \;\; wx_1 + px_2 \leq w.L + \pi(p, w).$

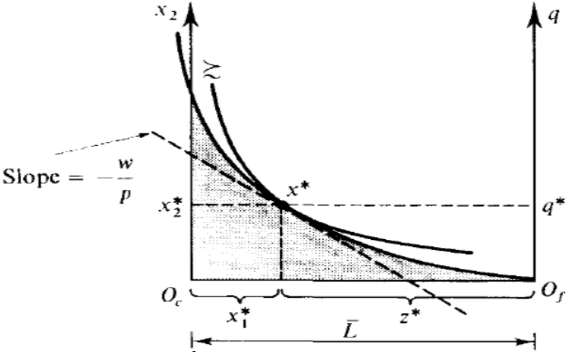


Figure 7: Failure of the 2^{ed} welfare theorem with non-convex technology.

The Second Fundamental Theorem of Welfare Economics

- A two consumers Edgworth box economy in which preferences are not convex.

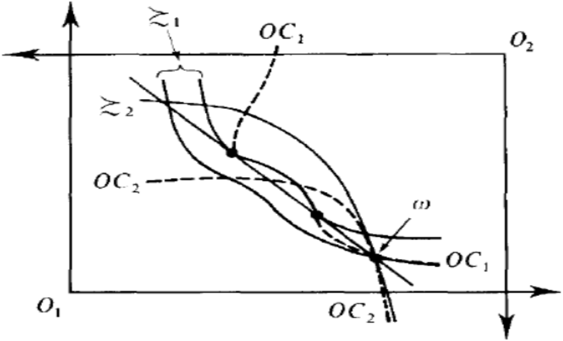


Figure 8: Failure of the 2^{ed} welfare theorem with non-convex preferences.

The Second Fundamental Theorem of Welfare Economics

- A two consumers Edgworth box economy in which preferences are convex, but the Pareto optimal allocation (ω_1, ω_2) cannot be supported as a price equilibrium with transfers, $p = (p_1, \ldots, p_L) \gg \mathbf{0}$ and $\sum_i T_i = 0$.

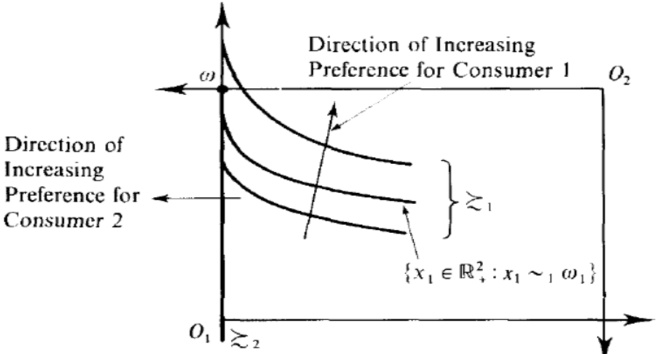


Figure 9: Failure of the 2^{ed} welfare theorem with non-convex preferences.

The Second Fundamental Theorem of Welfare Economics

- the endowment point is on the north west corner, consumer 1 posses all ω_2 and consumer (2) has all good ω_1.
- Indifference curves of consumer (2) are vertical and she desires only good 1.
- Consumer (1) would like to have more x_1, but consumer (2) never inclines for trade.
- If relative prices is $\frac{p_2}{p_1} > 0$ then consumer 2's optimal demand is his endowment, but consumer 1's initial endowment is never her optimal demand in any relative prices $\frac{p_2}{p_1} > 0$.
- In any $\frac{p_2}{p_1} > 0$ consumer 1 wishes to buy strictly positive amount of x_1.
- Consumer's (1) demand for good 2 is infinite when $\frac{p_2}{p_1} = 0$

The Second Fundamental Theorem of Welfare Economics

- the figure 9 illustrates a type of failure of supportability by means of prices
  - Both consumers have convex preferences
  - the corner solution is Pareto allocation
  - but the allocation (ω_1, ω_2) can not be supported as a price equilibrium with transfers;

- the ω_2 is an optimal demand for the consumer 2 for any price vector $p = (p_1, p_2) \geq 0$.

- the ω_1 is an optimal demand for consumer 1 for ***no*** price vector $p \geq 0$ and wealth level w_1

- to tackle the problem one can establish a version of the 2^{nd} welfare theorem that allows the failure.

The Second Fundamental Theorem of Welfare Economics

- This is done by defining the concept of a ***price quasi-equilibrium with transfers***, a weakening form of price equilibrium with transfers.

- the definition of ***price quasi-equilibrium with transfers*** is identical to that of ***price equilibrium with transfers*** except that the condition[if $x_i \succ_i x_i^*$], then $p.x_i > w_i$ is replaced with $x_i \succ_i x_i^*$, then $p.x_i \geq w_i$].

The Second Fundamental Theorem of Welfare Economics

Definition

price quasi-equilibrium with transfers; Given an economy specified by $(\{(\succsim_i, X_i)\}_{i=1}^{I}, \{Y_j\}_{j=1}^{J}, \{\bar{\omega}\})$ an allocation (x^*, y^*) and price vector $p = (p_1, ..., p_L) \neq 0$ constitute a *price quasi-equilibrium with transfers* if there is an assignment of wealth levels $(w_1, ..., w_I)$ with $\sum_i w_i = p.\bar{\omega} + \sum_j p.y_j^*$ such that

1. For every j, y_j^* maximizes profit in Y_j; that is
$$p.y_j \leq p.y_j^* \text{ for all } y_j \in Y_j$$
2. For every i, if $x_i \succ_i x_i^*$ then $p.x_i \geq w_i$.
3. $\sum_i x_i^* = \bar{\omega} + \sum_i y_j^*$

The Second Fundamental Theorem of Welfare Economics

Theorem

The Second Fundamental Theorem of Welfare Economics; Consider an economy specified by $(\{(\succsim_i, X_i)\}_{i=1}^{I}, \{Y_j\}_{j=1}^{J}, \{\bar{\omega}\})$ *and suppose that every* Y_j *is convex and every preference relation* $\succsim_i$ *is convex [i.e. the upper contour set is convex] and locally non-satiated. then, for every Pareto optimal allocation* (x^*, y^*)*, there is a price vector* $p = (p_1, ..., p_I) \neq 0$ *such that* (x^*, y^*, p) *is a price quasi-equilibrium with transfers.*

Definition

For every i, the set V_i of consumptions preferred to x_i^*, that is, $V_i = \{x_i \in X_i : x_i \succ_i x_i^*\} \subset \mathbf{R}^L$. then define

$$V = \sum_i V_i = \{\sum_i x_i \in \mathbf{R}^L : x_1 \in V_1, ..., x_I \in V_I\}$$

and

$$Y = \sum_j Y_j = \{\sum_j y_j \in \mathbf{R}^L : y_1 \in Y_1, ..., y_J \in Y_J\}$$

The Second Fundamental Theorem of Welfare Economics

- Thus, V is the set of aggregate consumption boundless, which each of them is strictly preferred to x_i^*.
  - Example: $A = [0, 1], B = \{1\}$ then $A + B = [1, 2]$ but $A \cup B = [0, 1]$
- The set Y is simply the aggregate production set
- And the $Y + \{\bar{\omega}\}$ is the aggregate production set with its origin shifted to $\bar{\omega}$.

Proof.

1. Every set V_i is convex. Proof by contradiction, In class.
2. The set V and $Y + \{\bar{\omega}\}$ are convex set. Proof by the definition of aggregated sets, In class.

□

Proof.

Let A, B be any two convex subsets of $\mathbb{R}^L$. That is, for any element x, y of A (resp. of B), and any real scalar t in [0,1], the "convex combination":

$$tx + (1 - t)y$$

again belongs to A (resp. to B).

First we will show that the set $A + B$ (formed by taking all terms $a + b$ where $a \in A$, $b \in B$) is again a convex set. For suppose $x + y$ and $x' + y'$ are any two points in $A + B$, naturally with x, x' in A and y, y' in B. Given real t in [0,1], we know:

$\square$

Proof.

Conti.

$$
\begin{aligned}
tx + (1-t)x' \in A \\
ty + (1-t)y' \in B
\end{aligned}
$$

and therefore:

$$tx + (1-t)x' + ty + (1-t)y' = t(x+y) + (1-t)(x'+y')$$

will also belong to $A + B$. Therefore $A + B$ is shown to be convex. $\square$

The Second Fundamental Theorem of Welfare Economics

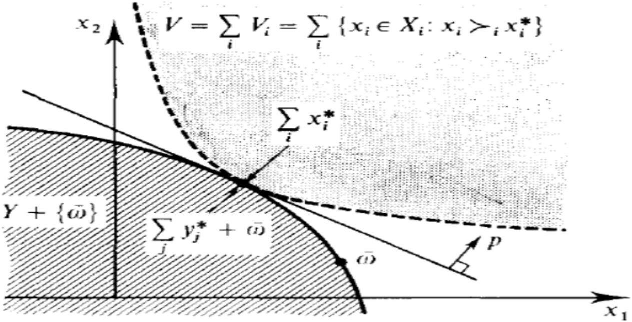


Figure 10: two separated sets

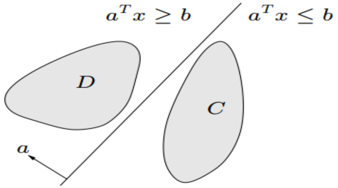
Proof.

3. $V \cap (Y + \{\bar{\omega}\}) = \varnothing$. Proof by the definition of Pareto optimality and the sufficient condition of the theorem, Proof in class.

Separating hyperplane theorem

if C and D are nonempty disjoint convex sets, there exist $a \neq 0$, b s.t.

$$a^T x \leq b \text{ for } x \in C, \qquad a^T x \geq b \text{ for } x \in D$$



the hyperplane $\{x \mid a^T x = b\}$ separates C and D

strict separation requires additional assumptions ($e.g.$, C is closed, D is a singleton)

Figure 11: Separating hyperplane theorem, $C \cap D = \varnothing$

The Second Fundamental Theorem of Welfare Economics

Proof.

4. There is $p = (p_1, ..., p_L) \neq 0$ and a number r such that $p.z \geq r$ for every $z \in V$ and $p.z \leq r$ for every $z \in Y + \{\bar{\omega}\}$,Proof in class.

5. If $x_i \succsim_i x_i^*$ for every i then $p.(\sum_i x_i) \geq r$.

Definition

Open Sets and Limit Points: A point xx is a limit point of a set AA iff there exists a sequence (an)?A(an)?A, satisfying an?xan?x, that converges to xx. Note that xx need not be a point in AA. where $AA = [0, 1)$ then,

$$\lim_{n \to \infty} x_n = \lim_{n \to \infty} (1 - 1/n)$$

a point for which every neighbourhood contains at least one point belonging to a given set.

The Second Fundamental Theorem of Welfare Economics

Proof.

6. $p.(\sum_i x_i^*) = p.(\bar{\omega} + \sum_j y_j^*)$.

7. For every j, we have $p.y_j \leq p.y_j^*$ for all $y_j \in Y_j$.

8. For every i, if $x_i \succ_i x_i^*$, then $p.x_i \geq p.x_i^*$.

9. The wealth levels $w_i = p.x_i^*$ for $i = 1, ..., I$ support (x^*, y^*, p) as a price quasi-equilibrium with transfer.

Conditions (i) and (ii) of definition (16.D.1) follow from steps 7 and 8, condition (iii) follows the feasibility of the Pareto optima allocation (x^*, y^*).■

□

The Second Fundamental Theorem of Welfare Economics: Some of practical limitations

- The Second Fundamental Theorem of Welfare Economics identifies conditions under which any Pareto optimal allocation can be implemented throgh competitive markets.
- But there are some practical limitations on the use of this theoretical result.
- **First**, a central planner must be able to insure that the supporting prices $(p_1, ..., p_L)$ will be taken as given by consumers and firms.
- If the market structure is such that price taking would not automatically hold, then the planer must be able to enforce it.

The Second Fundamental Theorem of Welfare Economics: Some of practical limitations

- A **Second observation** is that the central planer must have very good information indeed.
- It must have good information to Identify the Pareto optimal allocation to be implemented and it must have the ability to tell who is who by observing her preferences and endowments perfectly.
- Such information is rarly available for him in practice, as a result he has to make the transformation in lump-sum from.
- Finally, even if he has all the required information, it must actually have the power to enforce the necessary wealth transfers through a tax mechanism that individuals can not evade.

Pareto Optimality and Social Welfare Optima

- This section discuss the relationship between pareto optimality and optimization ofa social welfare function.
- The utility possibility set: $U = \{(u_1, ..., u_I) \in \mathbf{R}^I$: there is a feasible allocation(x, y) such that $u_i \leq u_i(x_i)$ for $i = 1, ..., I\}$.

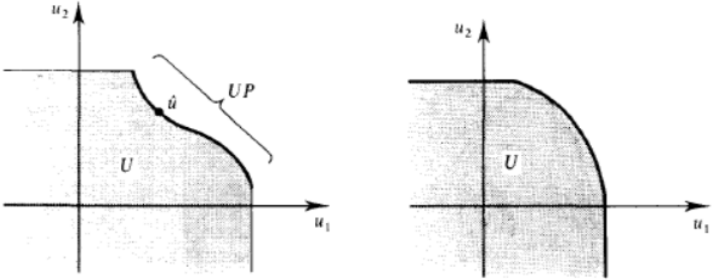


Figure 12: the utility possibility set and a convex utility possibility set. If every X_i and every Y_j is convex set, and if each$\succsim_i$ is convex, then U is a convex set

Pareto Optimality and Social Welfare Optima

- Pareto Frontier UP, Definition:

Definition

By the definition of Pareto optimality, the utility values of a Pareto optimal allocation must belong to the boundary of the utility set.
$UP = \{(u_1, ..., u_I) \in U\colon$ there is no $(u_1', ..., u_I') \in U$ such that $u_i' \geq u_i$ for all i and $u_i' > u_i$ for some $i\}$.

- then the following proposition is intuitive.

Theorem

A feasible allocation $(x, y) = (x_1, ..., x_I, y_1, ..., y_J)$ *is a Pareto optimum if only if* $(u_1(x_1), ..., u_I(x_I)) \in UP$.

Social welfare function and distributional concerns

- Suppose that society's distributional principles are summarized in a ***social welfare function*** $W(u_1, ..., u_I)$.

- a simple form of the $W(.)$ is the linear one, however, the **Rawlsian** social welfare function which very popular in the distributional issues is of the following form:

$$W(u_1, ..., u_I) = Min\{u_1, ..., u_I\}$$

- The linear form of the **SWF** is;

$$W(u_1, ..., u_I) = \sum_i \lambda_i u_i$$

  where the λ_i are some $\lambda_i \geq 0$ constants.

- With this Linear **SWF**, a feasible optimal value for u_i is resulted from constrained maximization, $\underset{u \in U}{Max} \lambda.u$

Social welfare function and distributional concerns

- a social distribution depends on the λs.

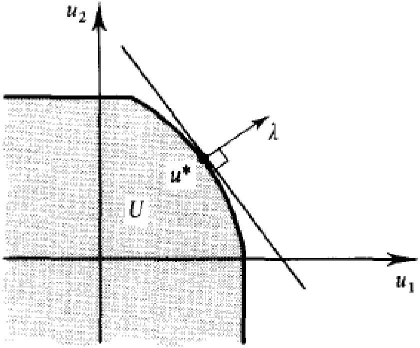


Figure 13: maximizing a linear social welfare function.

Pareto Optimality and Social Welfare Optima

Figure 13 depicts the solution to problem $\underset{u \in U}{Max} \lambda.u$. The result is presented in following theorem:

Theorem

- *If* $u^* = (u_1^*, ..., u_I^*)$ *is a solution to the social welfare maximization problem* $\underset{u \in U}{Max} \lambda.u$ *with* $\lambda \gg 0$*, then* $u^* \in UP$*, that is* u^* *is the utility vector of a Pareto optimal allocation.*

- *Moreover, if the utility possibility set* U *is convex, then for any* $\tilde{u} = (\tilde{u}_1, ..., \tilde{u}_I) \in UP$*, there is a vector of welfare weights* $\lambda = (\lambda_1, ..., \lambda_I) \geq 0, \lambda \neq 0$*, such that* $\lambda.\tilde{u} \geq \lambda.u$ *for all* $u \in U$*, that is, such that* $\tilde{u}$ *is a solution to the social welfare maximization problem* $\underset{u \in U}{Max} \lambda.u$*.*

Pareto Optimality and Social Welfare Optima

Proof.

- **Proof by Contrapositive**: If $u^* = (u_1^*, ..., u_I^*)$ were not Pareto optimal, then there would exist a $u \in U$ with $u \geq u^*$; and so because $\lambda \gg 0$, we would have $\lambda.u > \lambda.u^*$.

- **Proof by the supporting hyperplane theorem**. Note that if $\tilde{u} \in UP$, then $\tilde{u}$ is boundary of U. By the supporting hyperplane theorem there exist a $\lambda \neq 0$ such that $\lambda.\tilde{u} \geq \lambda.u$ for all $u \in U$.

□

Social welfare function and distributional concerns

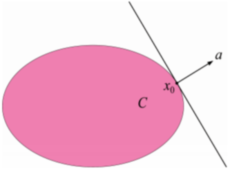
Supporting Hyperplane Theorem

Th. Let $C \subseteq \mathbb{R}^n$ be a *nonempty convex* set. Let x_0 be such that

either $x_0 \in \mathrm{bd}C$ **or** $x_0 \notin C$

Then, there exists a hyperplane passing through x_0 and containing the set C in one of its halfspaces, i.e., there is a vector $a \in \mathbb{R}^n$, $a \neq 0$, such that

$$\sup_{z \in C} a^T z \leq a^T x_0$$



- A hyperplane with such property is referred to as a *supporting hyperplane*

Figure 14: the supporting hyperplane theorem.

linear Social welfare function and Pareto optima

- The just proved theorem tells us that for economies with convex utility possibility sets, there is a close relation between Pareto optima and linear social welfare optima.

- Every linear social welfare optimum with $\lambda \gg 0$ is Pareto optimal, and also the reverse is true for some $(\lambda_1, ..., \lambda_I) \geq 0$.

First-Order conditions for Pareto optimality

- How prices and the optimality properties of price taking behavior emerge from an examination of the first order conditions associated with Pareto optimality problem?

- some assumptions;
  - $u_i(x_i)$ are twise continuously differentiable and satisfy $\nabla u_i(x_i) \gg 0$ at all x_i, suggesting that $\succsim_i$ is strongly monotone.

  - For every j the production set is $Y_j = \{y \in \mathbf{R}^L : F_j(y) \leq 0\}$, where $F_j(y) = 0$ defines the production frontier.

  - $\nabla F_j(y_j) = (\partial F_j(y_j)/\partial y_{1j}, ..., \partial F_j(y_j)/\partial y_{Lj}) \gg 0$ for all $y_j \in \mathbf{R}^L$.

  - what does this assumption mean?

First-Order conditions for Pareto optimality

- The problem of Pareto optimal allocation identification to select $(x, y) = (x_1, ...x_I, y_1, ..., y_J) \in \mathbf{R}_+^{LI} \times \mathbf{R}^{LJ}$

$$Max\ u_1(x_{11}, ..., x_{L1}) \tag{1}$$

$$s.t.\ u_i(x_{1i}, ..., x_{Li}) \geq \bar{u}_i \quad i = 1, ..., I \tag{2}$$

$$\sum_i x_{li} \leq \omega_l + \sum_j y_{lj} \qquad l = 2, ..., L \tag{3}$$

$$F_j(y_{1j}, ..., y_{Lj}) \leq 0 \qquad j = 1, ..., J \tag{4}$$

$$\mathcal{L} = u_1(x_{11}, ..., x_{L1}) + \sum_{i=2} \delta_i[u_i(x_{1i}, ..., x_{Li}) - \bar{u}_i]$$
$$+ \sum_l \mu_l[\omega_l + \sum_j y_{lj} - \sum_i x_{li}] +$$
$$\sum_j \gamma_j[0 - F_j(y_{1j}, ..., y_{Lj})]$$

First-Order conditions for Pareto optimality

$$\partial\mathcal{L}/\partial x_{li} = \partial u_i(x_i)/\partial x_{li} - \mu_l \leq 0 \tag{5}$$

$$\partial\mathcal{L}/\partial x_{l'i} = \partial u_i(x_i)/\partial x_{l'i} - \mu_{l'} \leq 0 \tag{6}$$

$$i = 1, l = 1, 2, ..., L, l \neq l' \tag{7}$$

$$\partial\mathcal{L}/\partial x_{li} = \delta_i \partial u_i(x_i)/\partial x_{li} - \mu_l \leq 0 \tag{8}$$

$$\partial\mathcal{L}/\partial x_{l'i} = \delta_i \partial u_i(x_i)/\partial x_{l'i} - \mu_{l'} \leq 0 \tag{9}$$

$$i = 2, ..., I, l = 1, 2, ..., L, l \neq l' \tag{10}$$

$$\partial\mathcal{L}/\partial y_{lj} = \mu_l - \gamma_j \partial F_j(y_j)/\partial y_{lj} \leq 0 \tag{11}$$

$$\partial\mathcal{L}/\partial y_{l'j} = \mu_{l'} - \gamma_j \partial F_j(y_j)/\partial y_{l'j} \leq 0 \tag{12}$$

$$\partial\mathcal{L}/\partial y_{l'j'} = \mu_{l'} - \gamma_{j'} \partial F_{j'}(y_{j'})/\partial y_{l'j'} \leq 0 \tag{13}$$

$$\partial\mathcal{L}/\partial y_{lj'} = \mu_l - \gamma_{j'} \partial F_{j'}(y_{j'})/\partial y_{lj'} \leq 0 \tag{14}$$

$$j = 1, 2, ..., J, l = 1, 2, ..., L, l \neq l', j' \neq j \tag{15}$$

First-Order conditions for Pareto optimality

- The value of μ_l at optimal solution is exactly the increase in the utility of consumer 1 from marginal increase in the available social endowment $\bar{\omega}_l$, Equations (5) and (6).
- δ_i reflects the marginal change in the consumer's 1 utility when a decrease in the $u_i, i = 2, 3, ..., I$ is realized. Recall that $\delta_i = \frac{\partial u_1}{\partial u_i}, ; i > 1$
- Therefore conditions (8) and (9) show that, at an interior optimal allocation, the increase in the utility of any consumer i from receiving an additional unit of good l, weighted consumer i's utility constraint is worth in terms of raising consumer 1's utility, should be equal to the marginal value μ_l of good l. That is
$$\frac{\partial u_1}{\partial u_i}\frac{\partial u_i}{\partial x_{li}} = \mu_l.$$

First-Order conditions for Pareto optimality

- The multiplier γ_j is the marginal benefit from relaxing the jth production constraint, or the marginal cost of tightening it. Recall that $\gamma_j = \frac{\partial u_1}{\partial F_j}, ; j = 1, ..., J$

- Hence, $\gamma_j(\partial F_j / \partial y_{li})$ is the marginal cost of increasing in y_{lj}.
$$\frac{\partial u_1}{\partial F_j} \frac{\partial F_j}{\partial y_{li}} = \mu_l$$

- at the optimum of u_1 this marginal cost is equated, for every j, to the marginal benefit μ_l of good l.

- If the solution for the first order conditions is interior, $x_i \gg 0, y_j \gg 0$, then the conditions (5)-(15) will give us;

First-Order conditions for Pareto optimality

$$\frac{\partial u_i/\partial x_{li}}{\partial u_i/\partial x_{l'i}} = \frac{\partial u_i/\partial x_{li'}}{\partial u_{i'}/\partial x_{l'i'}} = \mu_l/\mu_{l'} \;\; \forall\, i.i', l, l' \tag{16}$$

$$\frac{\partial F_j/\partial y_{lj}}{\partial F_j/\partial y_{l'j}} = \frac{\partial F_j/\partial y_{lj'}}{\partial F_{j'}/\partial y_{l'j'}} = \mu_l/\mu_{l'} \;\; \forall\, i.i', l, l' \tag{17}$$

$$\frac{\partial u_i/\partial x_{li}}{\partial u_i/\partial x_{l'i}} = \frac{\partial F_j/\partial y_{lj}}{\partial F_j/\partial y_{l'j}} = \mu_l/\mu_{l'} \;\; \forall\, i.i', l, l' \tag{18}$$

- The condition (18) says that every consumer's marginal rate of substitution must equal every firm's marginal rate of transformation for all pairs of goods.
- Imagine an economy with 2 goods, one consumer and one firm

Relationship between the first order conditions and the first and second welfare theorems

- preference relations are convex
- production sets are convex
- Conditions (5)-(15) are used to established a version of the two theorems.
- Let (x^*, y^*, p) is a price equilibrium with transfers IFF the first order condition for utility maximization, profit maximization and a linear social welfare function are satisfied.

$$Max \;\; u_i(x_i) \tag{19}$$

$$s.t. \;\; p.x_i \leq w_i \tag{20}$$

$$Max \;\; p.y_j \tag{21}$$

$$s.t. \;\; F_j(y_j) \leq 0 \tag{22}$$

Relationship between the first order conditions and the first and second welfare theorems

- Denoting by α_i and β_j the respective multipliers for the optimization problem, the FOC is derived as follows:

$$x_{li} : \frac{\partial u_i}{\partial x_{li}} - \alpha_i p_l \leq 0 \ \ \forall i, l \tag{23}$$

$$y_{lj} : p_l - \beta_j \frac{\partial F_j}{\partial y_{lj}} \leq 0 \ \ \forall j, l \tag{24}$$

- Letting $\mu_l = p_l, \delta_i = 1/\alpha_i$ and $\gamma_j = \beta_j$

Maximization of social welfare function subject to social resources

- In a linear social welfare function setting one can show that optimal solution for a maximization problem corresponds to Pareto optimum allocation.
- Consider the problem:

$$Max_{x,y} \sum_i \lambda_i u_i(x_{1i}, ..., x_{1i}) \tag{25}$$

$$s.t.(1) \sum_i x_{li} \leq \bar{\omega}_l + \sum_j y_{lj} \qquad l = 1, ..., L \tag{26}$$

$$(2) F_j(y_{1j}, ..., y_{Lj}) \leq 0 \qquad l = 1, ..., L \tag{27}$$

Maximization of social welfare function subject to social resources, FOC

- The first order condition for the optimization program is:

$$Max_{x,y}\mathcal{L} = \sum_i \lambda_i u_i(x_{1i}, ..., x_{1i}) + \sum_l \psi_l[\bar{\omega}_l + \sum_j y_{lj} - \sum_i x_{li}] \tag{28}$$

$$+ \sum_l \eta_l[0 - F_j(y_{1j}, ..., y_{Lj})] \tag{29}$$

$$x_{li} : \lambda_i \frac{\partial u_i}{\partial x_{li}} - \psi_l \leq 0 \ \ \forall i, l \tag{30}$$

$$y_{lj} : \psi_l - \eta_j \frac{\partial F_j}{\partial y_{lj}} = 0 \ \ \forall j, l \tag{31}$$

- By letting $\delta_i = \frac{\lambda_i}{\lambda_1}$, $\mu_l = \frac{\psi_l}{\lambda_1}$ and $\gamma = \frac{\eta_j}{\lambda_1}$, we have and exact correspondence between FOC in pareto optimality program and the FOC above.

Theorem

Under the assumptio made about the economy, including, the concavity of $u_i(.)$ and the convexity of $F_j(.)$, every Pareto optimal allocation and price equilibrium with transfers, maximizes a weighted sum of utilities s.t. te resources and technologies.

Some applications, interpretations of Commodity Space

- **Contingent commodities**
  - One commodity in uncertain states
- **Occupational Choice**
  - Two person with consumption goods and labor supply for two alternative jobs, classic scholar and economics professors.
- **Public Goods, Lindahl Equilibrium**
  - One private good (labour) and a public good. The private good is used to produce the public good.
  - The problem of missing market for public goods is solved in the framework of Pareto allocation as a personalized good with market prices p_{2i}.
  - The resulted price equilibrium still is Pareto optimal, and there is a vector of prices which supports the Pareto equilibrium.

Public Goods, Lindahl Equilibrium, a formal representation

- a private good, x_{1i} which is tradable in the market forp_{1i}
- a public good which at the first look does not seem to be tradable in the market, but one can modify the Walrasian equilibrium to find a market solution for its allocation between users of the good. **HOW?**
- consumers' locally non-satiated preferences $\succsim_i$ are defined on consumption set $\mathbf{R}^2_+$.

Public Goods, Lindahl Equilibrium, a formal representation

- the economy is only endowed with $\bar{\omega}_1$ and a firm which transforms amounts of z, the private good, into the public good by making use of a concave production technology $f(z)$.
- An allocation $((x_{11}, ..., x_{1I}, x_2), (q, z) \geq 0$ is feasible if

$$q \leq f(z), \sum_i x_{1i} + z = \bar{\omega}_1)$$

  and $q = x_2$.
- A typical consumer cares about the amount of personal commodity that she receives, therefore we denote her consumption bundle by $x_i = (x_{1i}, x_{2i})$.

Public Goods, Lindahl Equilibrium, a formal representation

- The single firm's convex production (technology) set is:

  $$Y = \{(-z, q_1, ..., q_I) \in \mathbf{R}_+^{I+1} : z \geq 0 \text{ and } q_1 = ... = q_I = q \leq f(z)\}$$

- **A Lindahl Equilibrium**