Microeconomics I for Ph.D.
Chapter one: Social Choice Theory

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Outline

Social preferences over two alternatives

The general case: Arrow's impossibility theorem

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Some possibility Results: Restricted Results

Less than Full Social Rationality

introduction

- There are two alternatives x and y over which we begin our analysis
- There are $I < \infty$ of individuals or agents
- Individual's preference over the alternatives is shown by $\alpha_i \in \{-1, 0, 1\}$
  where α_i takes the value 1, 0, or -1 respectively if agent i prefers x toalternative y, is indifferent between two alternatives or prefers y over x.
- The family of individual preferences between the two alternatives can be described by a profile $(\alpha_1, ..., \alpha_I) \in \Re^I$

Example

let $I = 3$ and $\Im = \{(-1,-1,-1), (-1,-1,0), ..., (1,1,1)\}$ and $\#\Im = 3^3$

Social welfare functional

Functional

In mathematics, and particularly in functional analysis and the calculus of variations, a is a function from a vector space into its underlying field of scalars.

Definition

Social Welfare Functional: A social welfare functional is a rule $F(\alpha_1, ..., \alpha_I)$ that assigns a social preference, that is $F(\alpha_1, ..., \alpha_I) \in \{-1, 0, 1\}$, to every possible profile of individual preferences $(\alpha_1, ..., \alpha_I) \in \{-1, 0, 1\}^I$.

Definition

Paretian Social Welfare Functional: A social welfare functional is a rule $F(\alpha_1, ..., \alpha_I)$ has the Pareto property, if it respects unanimity of strict preference on the part of the agents, that is, if $F(1, ..., 1) = 1$ and $F(-1, ..., -1) = -1$.

Social welfare functional

Example

Social Welfare Functional: A social welfare functional is a rule $F(\alpha_1, ..., \alpha_I)$ that assigns a social preference, that is $F(\alpha_1, ..., \alpha_I) \in \{-1, 0, 1\}$, to every possible profile of individual preferences $(\alpha_1, ..., \alpha_I) \in \{-1, 0, 1\}^I$.

Definition

Let $(\beta_1, ..., \beta_I) \in^I$, but not all equal to zero.then we could define a S.W.F

$$F(\alpha_1, ..., \alpha_I) = sign(\sum_i \beta_i \alpha_i)$$

- majority voting $\beta_i = 1$ for all $i \in I$

Social welfare functional

Example

Dictatorship: We say that a social welfare functional is *dictatorial* if there is an agent h, called *dictator*, such that, for any profile$(\alpha_1, ..., \alpha_I), \alpha_h = 1$ implies $F(\alpha_1, ..., \alpha_I) = 1$ and similarly, $\alpha_h = -1$ implies $F(\alpha_1, ..., \alpha_I) = -1$. The strict preference of the dictator prevails as the social preference.

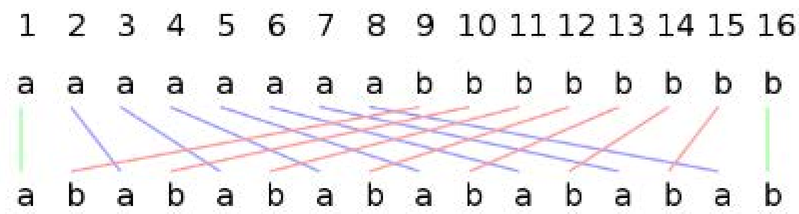
- This social welfare function is Paretian.
- we have dictatorship for the $F(\alpha_1, ..., \alpha_I) = sign(\sum_i \beta_i \alpha_i)$, if $\alpha_h > 0$ for some agent h and $\alpha_i = 0$ for some agent $i \neq h$, since then for $\beta_i = 1$ we have $F(\alpha_1, ..., \alpha_I) = \alpha_h$
- The majority voting rule has three more properties

Social welfare functional

- The majority voting rule has three more properties
  - symmetry among agents
  - neutrality between alternatives
  - positive responsiveness

Definition

Symmetry among agents: The social welfare functional $F(\alpha_1, ..., \alpha_I)$ is symmetric among agents(or anonymous) if a permutation of preferences across agents does not alter the social preference. Formally, let $\pi : \{1, ..., I\} \to \{1, ..., I\}$– [for instance $\pi(i) = I - i + 1$]– be an onto function with the property that for any i there is h such that $\pi(h) = i$. Then for any profile $(\alpha_1, ..., \alpha_I)$ we have $F(\alpha_1, ..., \alpha_I) = F(\alpha_{\pi(1)}, ..., \alpha_{\pi(I)})$.


1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16
a a a a a a a a b b b b b b b b
a b a b a b a b a b a b a b a b

Social welfare functional

Definition

Neutrality between alternatives: The social welfare functional $F(\alpha_1, ..., \alpha_I)$ is *neutrality between alternatives* if $F(\alpha_1, ..., \alpha_I) = -F(-\alpha_1, ..., -\alpha_I)$ for every profile $(\alpha_1, ..., \alpha_I)$. That is, if we reverse the preference of all agents, then the social preference is reversed.

Definition

Positively responsive: The social welfare functional $F(\alpha_1, ..., \alpha_I)$ is*positively responsive* if, whenever $(\alpha_1, ..., \alpha_I) \geqq (\alpha'_1, ..., \alpha'_I)$, $(\alpha_1, ..., \alpha_I) \neq (\alpha'_1, ..., \alpha'_I)$ and $F(\alpha'_1, ..., \alpha'_I) \geqq 0$, we have $F(\alpha_1, ..., \alpha_I) = +1$. That is, if x is socially preferred or indifferent to y and some agents raise their consideration of x, then x becomes socially preferred.

Social welfare functional

- Majority voting satisfies the three properties ***symmetry among agents***, ***neutrality between alternatives*** and ***positive responsiveness***.

Example

The majority voting between two alternatives satisfies the three properties

majority voting

- **Symmetry:** let $\pi : \{1, 2, ..., I\} \longrightarrow \{1, 2, ..., I\}$ be any permutation, then $\Sigma_{i=1}^{I} \alpha_i = \Sigma_{i=1}^{I} \alpha_{\pi(i)}$, which implies $Sign[\Sigma_{i=1}^{I} \alpha_i] = Sign[\Sigma_{i=1}^{I} \alpha_{\pi(i)}]$, which in turn implies that $F(\alpha_i, ..., \alpha_I) = F(\alpha_{\pi(i)}, ..., \alpha_{\pi(I)})$.

- **Neutrality:**
$F(\alpha_i, ..., \alpha_I) = Sign[\Sigma_{i=1}^{I} \alpha_i] = Sign[-\Sigma_{i=1}^{I} - \alpha_i] = -Sign[\Sigma_{i=1}^{I} - \alpha_i] = -F(-\alpha_i, ..., -\alpha_I)$.

- **Positive responsiveness:**Assume that $F(\alpha_i, ..., \alpha_I) \geqslant 0$, then $Sign[\Sigma_{i=1}^{I} \alpha_i] \geqslant 0$, which implies $\Sigma_{i=1}^{I} \alpha_i \geqslant 0$. Take $(\alpha' i, ..., \alpha'_I) \geqslant (\alpha_i, ..., \alpha_I)$ such that $(\alpha' i, ..., \alpha'_I) \neq (\alpha_i, ..., \alpha_I)$. Then , $\Sigma_{i=1}^{I} \alpha'_i > 0$, which implies that $Sign[\Sigma_{i=1}^{I} \alpha'_i] > 0$. Then by definition $F(\alpha'_i, ..., \alpha'_I) > 0$.

Dictatorship

Example

A dictatorship functional $F(\alpha_1, \alpha_I) = \alpha_1$ satisfies **neutrality** and **positive responsiveness** between two alternatives, **but it is not symmetric**.

May's Theorem

Theorem

***May's Theorem**: Asocial welfare function $F(\alpha_1, ..., \alpha_I)$ is a majority voting welfare functional if and only if it is symmetric among agents, neutral between alternatives, and positive responsive.*

- a bi-conditional logical statement
- if the social welfare functional is M.V.R, then the three conditions are satisfied (the necessity).
  We have already proved it in previous slides.
- if the three conditions are satisfied, then the social welfare functional is a M.V.R (the sufficiency)
  Proof in Class

The general case: Arrow's impossibility theorem

- Aggregating individual preferences over any number of alternatives.
- The set of alternatives is $\mathbf{X}$
- Let there are I agents, indexed $i = 1, ..., I$.
- Every agent i has a rational preference relation $\succsim_i$, defined on $\mathbf{X}$
- The strict preference and the indifference relation derived from $\succsim_i$ are denoted by $\succ_i$ and $\sim_i$ respectively.
  - if $x \succsim_i y$ but $\neg(y \succsim_i x)$, then $x \succ_i y$
  - if $x \succsim_i y$ and $y \succsim_i x$, then $x \sim_i y$

The general case: Arrow's impossibility theorem

- the set of all possible rational preference relations on $\mathbf{X}$ is denoted by $\mathcal{R}$ the set of all possible preference relations on $\mathbf{X}$ having the property that no two distinct alternatives are indifferent is denoted by $\mathcal{P}$.
- it is clear that $\mathcal{P} \subset \mathcal{R}$.
- In this section we focus on the largest domains, $\mathcal{A} = \mathcal{R}^I$ and $\mathcal{A} = \mathcal{P}^I$

Definition

A social welfare functional. A social welfare functional defined on a given subset$\mathcal{A} \subset \mathcal{R}^I$ is a rule $F : \mathcal{A} \to \mathcal{R}$ that assigns a rational preference relation $F(.) \in \mathcal{R}$, to any profile of individual rational preference relations $(\succsim_1, ..., \succsim_I)$ in the admissible domain $\mathcal{A} \subset \mathcal{R}^I$

The general case: Arrow's impossibility theorem

- For any profile $(\succsim_1, ..., \succsim_I)$, we denote by $F_p(\succsim_1, ..., \succsim_I)$ the strict preference relation derived from $F(\succsim_1, ..., \succsim_I)$. That is similar to the individual's indifferent preference relation, we let $xF_p(\succsim_1, ..., \succsim_I)y$ if $xF(\succsim_1, ..., \succsim_I)y$ holds but $yF(\succsim_1, ..., \succsim_I)x$ does not. Then we say that x ***is socially preferred to*** y.

- We read $xF(\succsim_1, ..., \succsim_I)y$ as x ***is socially as good as*** y.

The general case: Arrow's impossibility theorem

Definition

Paretian Social Welfare Functional The welfare functional $F : \mathcal{A} \to \mathcal{R}$ is Paretian if, for any pair of alternatives $\{x, y\} \subset \mathbf{X}$ and any preference profile $(\succsim_1, ..., \succsim_I) \in \mathcal{A}$, we have that x is socially preferred to y, that is, $xF_p(\succsim_1, ..., \succsim_I)y$, whenever $x \succ_i y$ for $\forall i \in I$.

Example

The Borda Count

- We assign a number of points $c_i(x)$ to every alternative x.
- $c_i(x) = n$ if x is the n^{th} ranked alternative in the ordering $\succsim_i$.

The general case: Arrow's impossibility theorem

- If some alternatives are ranked in the same order, then $c_i(x)$ is the average rank of the alternatives indifferent to x.
  - **Example:** if $X = \{x, y, z\}$ and $x \succsim_i y \sim_i z$ then $c_i(x) = 1$, and $c_i(y) = c_i(z) = (2+3)/2 = 2.5$.

- For any profile $(\succsim_1, ..., \succsim_I) \in \mathcal{R}^I$ the social ordering is determine by adding up the points. That is $x F_p(\succsim_1, ..., \succsim_I) y$ if $\Sigma_i c_i(x) < \Sigma_i c_i(y)$.

- **Intuition:** If x is preferred to y out of the choice set $\{x, y\}$ introducing a third option z, expanding the choice set to $\{x, y, z\}$, must not make y preferable to x. In other words, preferences for x or y should not be changed by the inclusion of z, i.e., z is irrelevant to the choice between x and y. This formulation appears in bargaining theory, theories of individual choice, and voting theory.

- Now the formal definition for **IIA** is given as follow:

The general case: Arrow?s impossibility theorem

Definition

Independence of irrelevant alternative: The social welfare functional $F : \mathcal{A} \rightarrow \mathcal{R}$ defined on the domain $\mathcal{A}$ satisfies the **IIA** if the social preference between any two alternatives $\{x, y\} \subset \mathbf{X}$ depends only on the profile of individual preferences over the same alternatives. Formally, for any pair of alternatives$\{x, y\} \subset \mathbf{X}$, and for any pair of preference profile $(\succsim_1, ..., \succsim_I) \in \mathcal{A}$ and $(\succsim'_1, ..., \succsim'_I) \in \mathcal{A}$ with the property that, for every i

$$x \succsim_i y \Leftrightarrow x \succsim'_i y$$
$$xF(\succsim_1, ..., \succsim_I)y \Leftrightarrow xF(\succsim'_1, ..., \succsim'_I)y$$

and

$$y \succsim_i x \Leftrightarrow y \succsim'_i x$$
$$yF(\succsim_1, ..., \succsim_I)x \Leftrightarrow yF(\succsim'_1, ..., \succsim'_I)x$$

Independence of irrelevant alternative

Example

The Borda Count: cont. We have that x preferred to y [in fact $c(x) = 3$] and $c(y) = 4$. For the preferences α' let consider.

$$x \succ'_1 y \succ'_1 z \; ; c_1(x) = 1, c_1(y) = 2, c_1(z) = 3$$
$$y \succ'_2 z \succ'_2 x \; ; c_2(x) = 3, c_2(y) = 1, c_2(z) = 2$$

we have that y is socially preferred over x.Therefore the example shows that the **IIA** condition is a strong restriction.

- Is there any way to proceed and the restriction is automatically satisfied?
  to some extent our answer is**YES**.
- Can we proceed in this way and end up with rational social welfare decision (aggregate) rule? the following example shows that it is real difficulty.

Independence of irrelevant alternative

Example

The Borda Count: the Borda count social welfare functional does not satisfy the **IIA**. Suppose there are two agents and three alternatives $\{x, y, z\}$. For the preferences

$$
\begin{array}{l}
x \succ_1 z \succ_1 y \Rightarrow c_1(x) = 1; c_1(z) = 2; c_1(z) = 3 \\
y \succ_2 x \succ_2 z \Rightarrow c_2(y) = 1; c_2(x) = 2; c_2(z) = 3
\end{array}
$$

and the social value for the welfare functional is:

$$
\begin{array}{c}
c_1(x) + c_2(x) = 1 + 2 \\
c_1(y) + c_2(y) = 3 + 1 \\
c_1(z) + c_2(z) = 2 + 3
\end{array}
$$

The Condoracet paradox

Figure: The voting paradox (also known as Condorcet's paradox or the paradox of voting) is a situation noted by the Marquis de Condorcet in the late 18th century, in which collective preferences can be cyclic (i.e., not transitive), even if the preferences of individual voters are not cyclic.

The Condoracet paradox

Example

Suppose that we were to try majority voting among and two alternatives, but are face with three alternatives.
Does this determine a social welfare functional? In general we run in to the Condoracet paradox in the following sense.
Let us see three alternatives $\{x, y, z\}$ and three agents.
The preferences of the three agents are:

$$x \succ_1 y \succ_1 z$$
$$z \succ_2 x \succ_2 y$$
$$y \succ_3 z \succ_3 x.$$

The Condoracet paradox, Cont.

- Then pairwise majority voting tells us that (given the situation of z) the alternative x must be socially preferred to y. Because agents 1 and 2 strongly prefer x over y, (namely $xF_p(\succsim_1, \succsim_2)y$).
- Similarly, y must be socially preferred to z, since agents 1 and 3 vote for$y$ against z, $yF_p(\succsim_1, \succsim_3)z$. By transitivity property of $F(.)$ we must have $xF_p(.)z$.
- On the other side agents 2 and 3 (as a majority in society with three agents) haves strong preference z over x, $zF_p(\succsim_2, \succsim_3)x$.

An introduction to Arrow's impossibility theorem

- Is the Condoracet paradox due to the any of the strong properties of majority voting, namely symmetry, neutrality and positive responsiveness?
- NO. The Paradox goes to the heart of the matter: **With pairwise independence** there are no social welfare functional, that satisfies a minimal form of symmetry among agents (**no Dictatorship**) and a minimal form of positive responsiveness (**Pareto property**).
- The next proposition is the central result of the chapter, the ***Arrow's impossibility theorem***.

Arrow's impossibility theorem

Figure: The impossibility theorem is named after economist Kenneth Arrow, who demonstrated the theorem in his doctoral thesis and popularized it in his 1951 book Social Choice and Individual Values. The original paper was titled "*A Difficulty in the Concept of Social Welfare*"

An introduction to Arrow's impossibility theorem

- when voters have more three than alternatives (options), no ranked order voting system can convert the ranked preferences of individuals into a community-wide (complete and transitive) ranking while also meeting a pre-specified set of criteria: non-dictatorship, Pareto efficiency, and independence of irrelevant alternatives.

Arrow's impossibility theorem

Example

Arrow's impossibility theorem the following shows the type of problem typical of an election. Consider the following example, where voters are asked to rank their preference of candidates x, y and z:

- 45 votes $x \succ y \succ z$ (45 people prefer x over y and prefer y over z)
- 40 votes $y \succ z \succ x$ (40 people prefer y over z and prefer z over x)
- votes $z \succ x \succ y$ (30 people prefer z over x and prefer x over y)
- Candidate x has the most votes, so he/she would be the winner. However, if y was not running, z would be the winner, as more people prefer z over x. (x would have 45 votes and z would have 70). This result is a demonstration of Arrow's theorem.

Arrow's impossibility theorem

Theorem

Suppose that the number of alternatives is at least three. Then every social welfare functional $F : \mathcal{A} \to \mathcal{R}$ which is ***Paretian*** *and satisfies* ***Pairwise independence*** *condition is* ***Dictatorial****.*

Definition

Decisive Given $F(.)$, we say that a subset of agents $S \subset I$ is:

1. **Decisive for**, x over y if whenever every agent in S prefers x to y and every agent not in the S prefers y to x, then x is **socially preferred** to y.
2. **Decisive if**, for any pair $\{x, y\} \subset \mathbf{X}$, then S is decisive for x over y, irrespective of $I \backslash S$.
3. **Completely Decisive** for x for y if whenever every agent in S prefers x to y, then x is **socially preferred** to y.

Arrow's impossibility theorem, Proof

The proof will proceed in 10 steps.

Lemma

Step 1: If for some $\{x, y\} \subset X, S \subset I$ *is decisive for* x *over* y*, then , for any alternative* $z \neq x$*,* S *is decisive for* x *over* z*. Similarly, for any* $z \neq y$*,* S *is decisive for* z *over* y*.*

Proof.

Assum that $z \neq y$. consider a profile of preferences $(\succcurlyeq_1, ..., \succcurlyeq_I) \in \mathcal{A}$, where

$$x \succ_i y \succ_i z \text{ for every } i \in S$$
$$\text{and}$$
$$y \succ_i z \succ_i x \text{ for every } i \in I \backslash S$$

Arrow's impossibility theorem, Proof, step 1, Cont.

- by decisiveness of S , $xF_p(\succsim_1, ..., \succsim_I)y$
- Since $y \succ_i z$ for every $i \in I$ and $F(.)$ satisfies the Pareto property, it follows that $yF_p(\succsim_1, ..., \succsim_I)z$.
- By transitivity of $F_p(.)$, we conclude that $xF_p(\succsim_1, ..., \succsim_I)z$.
- By the pairwise independence condition, it follows that x is socially preferred to z whenever every agents in S prefers x to z and every agent in $I \backslash S$ prefers z to x.
- S is ***Decisive for*** x over z.

Arrow's impossibility theorem, Proof, step 2

Lemma

Step 2: If for some $\{x, y\} \subset X, S \subset I$ *is decisive for* x *over* y *and* z *is a third alternative, then* S *is decisive for* z *over* w *and for* w *over* z*, where* $w \in X$ *is any alternative distinct from* z*.*

Proof.

- By step1, S is decisive for x over z and for z over y.
- By then, applying step 1 again, this time to the pair $\{x, z\}$ and the alternative w, we conclude that S is decisive for w over z.
- Similarly, applying step 1 to {z,y} and w, we conclude that S is decisive for z over w.

□

Arrow's impossibility theorem, Proof, step 3

Lemma

Step 3: If for some $\{x, y\} \subset X, S \subset I$ *is decisive for* x *over* y*, then* S *is decisive.*

Proof.

use step 2 and step 1, then conclude. □

Example

Here a few notations are introduced

- $I = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$
- $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$
- $T = \{7, 8, 9, 10, 11\}$

Arrow's impossibility theorem, Notation and Example

Example

- $S \backslash (S \cap T) = \{1,2,3,4,5,6,7,8\} \backslash (\{7,8\}) = \{1,2,3,4,5,6\}$
- $S \cap T = \{7,8\}$
- $T \backslash (S \cap T) = \{9,10\}$
- $I \backslash (S \cup T) = \{11\}$
- $[S \backslash (S \cap T)] \cup (S \cap T) = \{1,2,3,4,5,6,7,8\}$

Lemma

Step 4: If $S \subset I$ and $T \subset I$ are decisive, then $S \cap T$ is decisive.

Arrow's impossibility theorem

Proof.

Step 4: Take any triple of distinct alternatives $\{x, y, z\} \subset X$ and consider a profile of preferences $(\succsim_1, ..., \succsim_I) \in \mathcal{A}$ where

$$z \succ_i y \succ_i x \text{ for every } i \in S \backslash (S \cap T) \ \ldots (1)$$
$$x \succ_i z \succ_i y \text{ for every } i \in S \cap T \ \ldots\ldots.(2)$$
$$y \succ_i x \succ_i z \text{ for every } i \in T \backslash (S \cap T) \ldots (3)$$
$$y \succ_i z \succ_i x \text{ for every } i \in I \backslash (S \cup T) \ldots (4)$$

where: $[S \backslash (S \cap T)] \cup [S \cap T] \cup [T \backslash (S \cap T)] \cup [I \backslash (S \cup T)] = I$

Then:

- $zF_p(\succsim_1, ..., \succsim_I)y$ because $S (= [S \backslash (S \cap T)] \cup (S \cap T))$ is a decisive set.

□

Arrow's impossibility theorem

Proof.

Step 4: Cont.

- similarly, $xF_p(\succsim_1, ..., \succsim_I)z$ because $T(= [T\backslash(S \cap T)] \cup (S \cap T))$ is a decisive set, from (#2) and (#3).
- By transitivity of social welfare functional, we have $xF_p(\succsim_1, ..., \succsim_I)y$.
- By the pairwise independence (ignoring the position of alternative z) it follows that $S \cap T$ is decisive for x over y, because only every agents $i \in S \cap T$ strictly prefers x over y.

□

Lemma

Step 5: For any $S \subset I$, we have that either S or its complement, $I\backslash S$, is decisive.

Arrow's impossibility theorem

Proof.

Step 5: Take any triple of distinct alternatives $\{x, y, z\} \subset X$ and consider a profile of preferences $(\succsim_1, ..., \succsim_I) \in \mathcal{A}$ where:

$$x \succ_i z \succ_i y \text{ for every } i \in S \text{ ..... (1)}$$
$$y \succ_i x \succ_i z \text{ for every } i \in I \backslash S \ldots (2)$$

- There are two possibilities: either $xF_p(\succsim_1, ..., \succsim_I)y$, in which case by **PI**, S is decisive for x over y (hence by step 3, S is decisive).
- Or, $yF_p(\succsim_1, ..., \succsim_I)x$ and $I \backslash S$ is decisive.
- By the Paretian condition, we have $xF_p(\succsim_1, ..., \succsim_I)z$, (every $i \in I$ prefers x against y).
- Then by the transitivity property of $F(.)$ we have $yF_p(\succsim_1, ..., \succsim_I)z$.
- using **PI** condition again, we conclude that $I \backslash S$is desisive for y over z, (Step 3).

Arrow's impossibility theorem

Lemma

Step 6: If $S \subset I$ is decisive and $S \subset T$, then T is also decisive.

Proof.

Step 6:

- let $S \subset I$ is decisive, then $I \backslash S$ cannot be decisive.
- let T is not decisive, then $S \cap (I \backslash T) = \varnothing$ would be decisive. But no agent is in a empty set, nobody cannot from a decisive set. Consequently it follows from step 5 that T is decisive.
- Because of Pareto property the empty set of agents cannot be decisive, if no agent prefers x over y and every agent prefers y over x, then x is not socially preferred to y.

□

Arrow's impossibility theorem, step 7

Lemma

Step 7: let $S \subset I$ is decisive and it includes more than one agent, then there is a strict subset $S' \subset S$, $S' \neq S$, such that S' is decisive.

Proof.

Step 7:

- Take any $h \in S$. If $S \backslash \{h\}$ is decisive, then we have finished the proof.
- Suppose that $S \backslash \{h\}$ is not decisive. Then, by step 5, $I \backslash (S \backslash \{h\})$ is decisive.
- It follows from step 4, that $\{h\} = S \cap [(I \backslash S) \cup \{h\}]$ is also decisive.
- {h} is a strict subset of S.

Arrow's impossibility theorem, step 8

Lemma

Step 8: There is an $h \in I$ *such that* $S = \{h\}$ *is decisive.*

Proof.

Step 8:

- It follows from step 7, by taking $S \backslash \{h\}$ as a non decisive set and iterating the steps taken in step 7.

□

Arrow's impossibility theorem, step 9

Lemma

Step 9: If $S \subset I$ is decisive then, for any $\{x, y\} \subset X$ is completely decisive.

Arrow's impossibility theorem, step 10

Lemma

Step 10: If, for some $h \in I, S = \{h\}$ *is decisive, then* h *is a dictator.*

Arrow's impossibility theorem, What we have done?

- Steps 1 to 3 proved that if a subset of agents is decisive for some pair of alternatives, then it is decisive for all pairs.
- Steps 4 to 6 showed some algebraic properties of the family of decisive sets.
- Steps 7 and 8 used these to show that there is a smallest decisive set formed by a single agent.
- Steps 9 and 10 proved that this agent is a dictator.

Arrow's impossibility theorem

In short, the theorem states that no rank-order voting system can be designed that always satisfies these three "fairness" criteria:

- If every voter prefers alternative x over alternative y, then the group prefers x over y.
- If every voter's preference between x and y remains unchanged, then the group's preference between x and y will also remain unchanged (even if voters' preferences between other pairs like x and z, y and z, or z and w change).
- There is no "dictator": no single voter possesses the power to always determine the group's preference.

Some possibility Results: Restricted Results

- What Arrow shows is that we should not expect a collectivity of individuals to behave with the kind of coherence that we may hope from an individual.
- Arrow's theorem tells that the institutional detail and procedures of political process cannot be neglected. Recall the **M.V.R** with **IIA** for $X = \{x, y, z\}$ and $I = 3$ from the Condorcet paradox.

$$x \succ y \text{ for } i = 1$$
$$x \succ y \text{ for } i = 2$$
$$y \succ x \text{ for } i = 3$$

- alternative x is winner. Now we can vote for x against z.

$$x \succ z \text{ for } i = 1$$
$$z \succ x \text{ for } i = 2$$
$$z \succ x \text{ for } i = 3$$

- alternative z is winner.

Some possibility Results: Restricted Results, Cont.

- Now set the agenda on the taking up first y over z.

$$y \succ z \text{ for } i = 1$$
$$z \succ y \text{ for } i = 2$$
$$y \succ z \text{ for } i = 3$$

- alternative y is winner. Now we can vote for y against x.

$$x \succ y \text{ for } i = 1$$
$$x \succ y \text{ for } i = 2$$
$$y \succ x \text{ for } i = 3$$

- alternative x is winner.
- always the last alternative is the winner!
- To what extent we can escape the dictatorship conclusion if we relax some of the restrictions imposed by Arrow's theorem?

Can we scape the dictatorship conclusion?

- We can relax the rationality of **SWF**
- We will impose a restriction on the preference relations of agents, the ***single peakedness***.

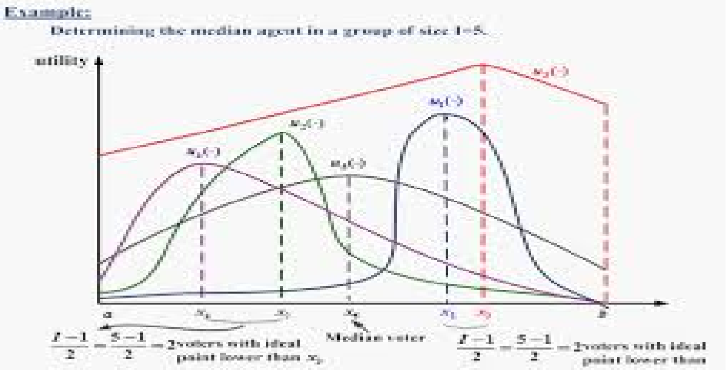


Figure: A group of agents is said to have single-peaked-preferences if: Each agent has an ideal choice in the set; and. For each agent, outcomes that are further from his ideal choice are preferred less.

Less than Full Social Rationality

- Suppose that we keep **PI**, Paretian conditions but let the **SWF** to be less tha fully rational.
- Two forms of weakening are made in social preference relation

Definition

Restricted Domain of preferences, Quasi-transitivity and Acyclicity of $\succsim$

Definition

Suppose that the preference relation $\succsim$ on X is reflexive and complete. we say then that:

1. $\succsim$ is quasi-transitive if the strict preference $\succ$ induced by $\succsim$ is transitive.
   - i.e. $x \succ y \Leftrightarrow x \succsim y$ but not $y \succsim x$
2. $\succsim$ is acyclic (transitive) if $\succsim$ has a maximal element in every finite subset $X' \subset X$, that is, $\{x \in X' : x \succsim y$ for all $y \in X\} \neq \varnothing$.

The social orderings of the Condorcet paradox also violates acyclicity, because it was not transitive.

Restricted Domain of preferences, Quasi-transitivity and Acyclicity of $\succsim$

Definition

Acyclicity: If, for all $x_1, x_2, \cdots, x_n \in X$,
$x_1 \succ x_2, \cdots, x_{n-1} \succ x_n$ implies $x_1 \succcurlyeq x_n$

Definition

Cyclicity: If, for all $x_1, x_2, \cdots, x_n \in X$,
$x_1 \succ x_2, \cdots, x_{n-1} \succ x_n$ implies $x_n \succcurlyeq x_1$

Example

Cyclicity:

Quasi-transitivity and Acyclicity of $\succsim$

Example

Suppose that X is a finite set of alternatives. Construct a reflexive and complete preference relation $\succsim$ on X with the property that $\succsim$ has a minimal element on every strict subset $X' \subset X$ and yet $\succsim$ is acyclic.

- Since by assumption X is finite, let $\#X = N$, then we can order its elements as $\{x_1, x_2, ..., x_n, \cdots, x_N\}$. Define the preference relation as follows:

1. For any $X' \subset X, x_i \in x_i$ implies $x_i \sim x_i$.
2. For any $X' \subset X$ such that $x_1 \notin X'$ or $x_N \notin X'$ then for all $\{x_i, x_j\} \subset X', x_i \succ x_j$ **iff** $j > i$.
3. For any $X' \subset X$ such that $x_1 \in X'$ and $x_N \in X'$ then $x_N \succ x_1$, and for all $\{x_i, x_j\} \subset X'$ such that $N \notin \{i, j\}, x_i \succ x_j$ **iff** $j > i$.

Quasi-transitivity and Acyclicity of $\succsim$, Cont

- This preference relation is complete and reflexive.
- For any strict subset of X it has a maximal element:
  1. For subset no including both x_1 and x_N then x_i with the smallest i is the maximal element.
  2. For subsets including x_1 and x_N then x_N is the maximal element. This preference is clearly complete, and it is reflexive.
- This preference is clearly complete, and it is reflexive as well. Because this can compare every element pair-wisely and by construction (1) every $x_i \sim x_i$.
- It has a maximal element on every strict subset of X.
- Finally this preference relation is not acyclic. WHY? see the definition above and item 3 in the previous slide.

Oligarchy Social Preference Relation

Example

Let I be the set of agents, and let $S \subset I$ be given subset of agents to be called an oligarchy (even $S = \{h\}$ or $S = I$). Given any profile $(\succsim_1, ..., (\succsim_I) \in \mathcal{R}^I$, the social preference are formed as follows:

Definition

For any $x, y \in X$, we say that x is socially at least as good as y $[xF(\succsim_1, ..., (\succsim_I)y]$ if there is at least one $h \in S$ that has $x \succsim_h y$. Hence, x is socially *referred* to y **iff** ***every*** member of the oligarchy prefers x to y.

- Oligarchy Social Preference Relation is quasitransitive, but not transitive (because social indifference fails to be transitive). This point will be discussed shortly, in the next slide.

Oligarchy Social Preference Relation

- This is Paretian
- This satisfies the **PI** condition

Example

Verify that the social preferences generated by the oligarchy are quasitransitive but that social indifference may not be transitive.

- If $xF_p(\succsim_1, ..., (\succsim_I)y$ and $yF_p(\succsim_1, ..., (\succsim_I)z$, this means that for all $h \in S$ members of oligarchy we have $x \succ_h y$ and $y \succ_h z$.
- Then transitivity of agents preference implies that $x \succ_h z$ for all $h \in S$, which results in $xF_p(\succsim_1, ..., (\succsim_I)z$.

Oligarchy Social Preference Relation

- To show that social indifference may not be transitive, let $I = \{1, 2, ..., I\}$ be the set of individuals and let $S = \{1, 2\}$ be the **Oligarchy**.
- Assume that the current preferences are $x \succ_1 z \succ_1 y$ and $y \succ_1 x \succ_1 z$.
- From the oligarchic social preference we have $xF_p(\succsim_1, ..., (\succsim_I)y$ and $yF_p(\succsim_1, ..., (\succsim_I)x$, so that x and y are socially indifferent.
- also we have $yF_p(\succsim_1, ..., (\succsim_I)z$ and $zF_p(\succsim_1, ..., (\succsim_I)y$, so that z and y are socially indifferent.
- However, since for both oligarchic agents $x \succ_1 z$ and $x \succ_2 z$, we have $xF_p(\succsim_1, ..., (\succsim_I)z$.
- We shown that social indifference is not transitive. This can happen as long as there are more than one member in the oligarchy.

Linear Order

- Single-peaked preference is the most important class of restricted domain condition. I this restricted domain, non-dictatorial aggregation is possible. On this domain pairwise majority voting gives rise to a social welfare functional.

Definition

Linear Order: A binary relation $\geq$ on the set of alternatives X is a linear order on X if it is **reflexive** (i.e., $x \geq x$ for every $x \in X$), **transitive** (i.e., $x \geq y$ and $y \geq z$ implies $x \geq z$) and **total** (i.e. for any distinct $x, y \in X$, we have that either $x \geq y$ or $y \geq x$, but not both).

Example

Let $X \subset \mathbb{R}$, then $\geq$ is a linear order on X.

Single-peaked

The rational preference relation $\succcurlyeq$ is ***single-peaked*** with respect to the linear order $\geq$ on X if there is an alternative $x \in X$ with property that $\succcurlyeq$ is increasing with respect to $\geq$ on $\{y \in X : x \geq y\}$ and decreasing with respect to $\geq$ on $\{y \in X : y \geq x\}$. That is,

$$\text{if } x \geq z > y \text{ then } z \succ y$$

and

$$\text{if } y > z \geq x \text{ then } z \succ y$$

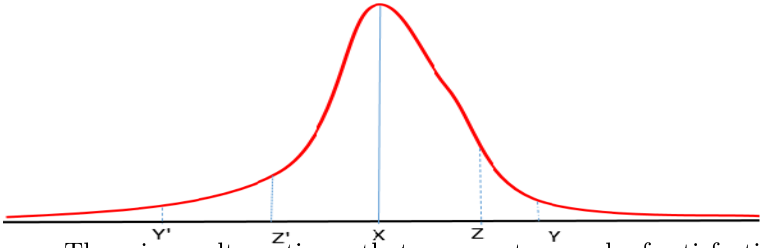


Figure: There is an alternative x that represents a peak of satisfaction and satisfaction increases as we approach this peak

Concavity of a single-peaked of $\succcurlyeq$

Example

Suppose that $X = [a, b] \subset \mathbb{R}$ and $\geqq$ is the GoE ordering of the real numbers. Then a continuous preference relation $\succcurlyeq$ on X is single peaked with respect to $\geqq$ **iff** it is strictly convex.

Proof.

Recall from the definition of a strictly convex $\succsim$ that:
for every $w \in X$, we have $\alpha y + (1 - \alpha)z \succ w$ whenever $y \succcurlyeq w, z \succcurlyeq w, y \neq z$ and $\alpha \in (0, 1)$.

- **The sufficiency:** suppose that x is the maximal element for $\succsim$ and $x > z > y$. Then $x \succsim y, y \succsim y$, and $z = \alpha x + (1 - \alpha)y$ for some $\alpha \in (0, 1)$. Thus, $z \succ y$ by strict convexity.

□

preferences of Condorcet paradox are not single-picked

Example

Show that the preferences of the Condorcet paradox are not single picked with respect to any possible linear order on the alternatives.

Solution: Suppose $X = \{x, y, z\}$, then the six possible orderings are: (1) (x, y, z); (2) (x, z, y); (3) (y, x, z); (4) (y, z, x); (5) (z, x, y); and (z, y, x). Recall that preferences' structure for Condorcet paradox were $x \succ_1 y \succ_1 z$, $z \succ_2 x \succ_2 y$, and $y \succ_3 z \succ_3 x$.

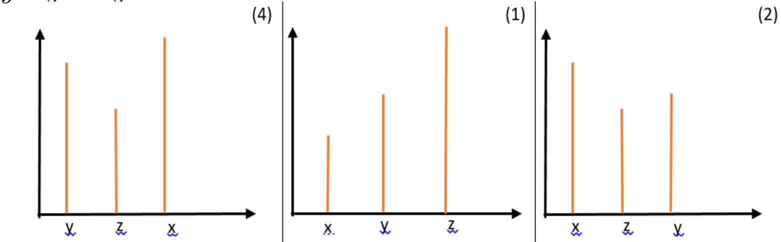


Figure: Agent 1's preferences are not single-picked for ordering (2) and (4), similarly orderings (2) and (4) for agent 2's preferences,...

Median agent

Definition

Median agent: Agent $h \in I$ is a median agent for the profile $(\succsim_1, ..., \succsim_I) \in \mathcal{R}^I_\geq$ if

$$\#\{i \in I : x_i \geq x_h\} \geq \frac{I}{2}$$
and
$$\#\{i \in I : x_i \leq x_h\} \geq \frac{I}{2}$$

A median agent always exists. If $\#I$ is odd, then $(I-1)/2$ of the agents have peaks strictly smaller than x_h and $(I-1)/2$ strictly larger than x_h. In this case median agent is unique.

Median agent

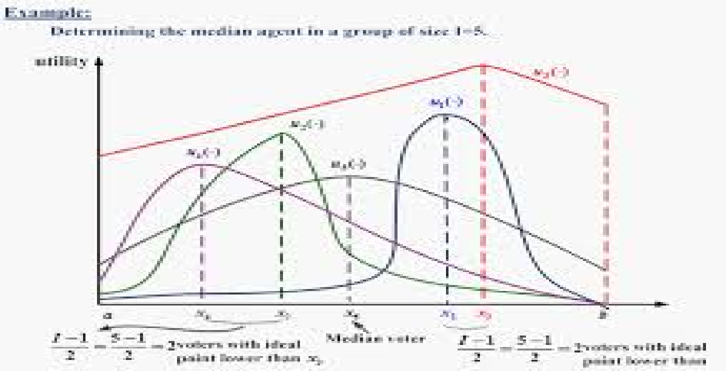


Figure: A group of agents is said to have single-peaked-preferences if: Each agent has an ideal choice in the set; and. For each agent, outcomes that are further from his ideal choice are preferred less.