11.B.4:

The first question to answer is that when we should have tax or subsidy? We know that we may use tax or subsidy when there is a source of externality. Here the source of externality is h. But what about e. Obviously the consumption of e doesn't have any effect on the utility of other players. Thus we would make things worse by having tax or subsidy on e. So no. We should not have tax or subsidy on e.

11.B.5:

a. We should write the profit maximization problem of the firm and use the Kuhn-Tucker conditions to write the F.O.C:

PMP: max $\pi = pq - c(q,h)$

$$\frac{\partial \pi}{\partial q} = p - \frac{\partial c(q,h)}{\partial q} \leq 0 \rightarrow \boldsymbol{p} \leq \frac{\boldsymbol{\partial c(q^*, h^*)}}{\boldsymbol{\partial q}} \quad w.eq\ if\ q^* > 0$$

$$\frac{\partial \pi}{\partial h} = 0 - \frac{\partial c(q,h)}{\partial h} \leq 0 \rightarrow \boldsymbol{0} \leq \frac{\boldsymbol{\partial c(q^*, h^*)}}{\boldsymbol{\partial h}} \quad w.eq\ if\ h^* > 0$$

b. To find the pareto optimal levels of goods, we need to maximize the total utility of the economy. In other words, we must maximize the sum of utility of consumer and the profit of the firm:

$Pareto\ optimal$: max $V = pq - c(q,h) + \phi(h) + w$

$$\frac{\partial V}{\partial q} = p - \frac{\partial c(q,h)}{\partial q} \leq 0 \rightarrow \boldsymbol{p} \leq \frac{\boldsymbol{\partial c(q^o, h^o)}}{\boldsymbol{\partial q}} \quad w.eq\ if\ q^o > 0$$

$$\frac{\partial V}{\partial h} = \phi'(h) - \frac{\partial c(q,h)}{\partial h} \leq 0 \rightarrow \boldsymbol{\phi'(h)} \leq \frac{\boldsymbol{\partial c(q^o, h^o)}}{\boldsymbol{\partial h}} \quad w.eq\ if\ h^o > 0$$

c. First assume that the government taxes the firm on output level. Thus the PMP for the firm is as follows:

PMP: max $\pi = pq - c(q,h) - tq$

$$\frac{\partial \pi}{\partial q} = p - t - \frac{\partial c(q,h)}{\partial q} \leq 0 \rightarrow \boldsymbol{p} \leq \frac{\boldsymbol{\partial c(q^*, h^*)}}{\boldsymbol{\partial q}} + \boldsymbol{t} \quad w.eq\ if\ q^* > 0$$

$$\frac{\partial \pi}{\partial h} = 0 - \frac{\partial c(q,h)}{\partial h} \leq 0 \rightarrow \ \mathbf{0} \leq \frac{\boldsymbol{\partial c(q^*, h^*)}}{\boldsymbol{\partial h}} \quad w.eq\ if\ h^* > 0$$

As we see, this kind of tax does not change the first order condition on h. Thus this kind of tax cannot help to restore efficiency.

Now let's assume that government taxes on h. Thus the PMP for the firm is as follows:

$$PMP\text{: max } \pi = pq - c(q,h) - th$$

$$\frac{\partial \pi}{\partial q} = p - \frac{\partial c(q,h)}{\partial q} \leq 0 \rightarrow \ \boldsymbol{p} \leq \frac{\boldsymbol{\partial c(q^*, h^*)}}{\boldsymbol{\partial q}} \quad w.eq\ if\ q^* > 0$$

$$\frac{\partial \pi}{\partial h} = -t - \frac{\partial c(q,h)}{\partial h} \leq 0 \rightarrow \ \boldsymbol{-t} \leq \frac{\boldsymbol{\partial c(q^*, h^*)}}{\boldsymbol{\partial h}} \quad w.eq\ if\ h^* > 0$$

If $t = -\phi'(h^*)$, then the answer is the same as optimal level. Thus a direct tax on the externality can restore efficiency.

d. Now assume that $h(q) = \alpha q$. Also assume that the government taxes the firm on output level.

$$PMP\text{: max } \pi = pq - c(q, \alpha q) - tq$$

$$\frac{\partial \pi}{\partial q} = p - \frac{\partial c(q,h)}{\partial q} - \frac{\partial c(q,h)}{\partial h}\frac{\partial h}{\partial q} \leq 0 \rightarrow \ \boldsymbol{p} \leq \frac{\boldsymbol{\partial c(q^*, h^*)}}{\boldsymbol{\partial q}} + \boldsymbol{\alpha} \frac{\boldsymbol{\partial c(q^*, h^*)}}{\boldsymbol{\partial q}} + \boldsymbol{t}$$

$$w.eq\ if\ q^* > 0$$

Assume that $q^* > 0$. So from above:

$$\boldsymbol{p} = \frac{\boldsymbol{\partial c(q^*, h^*)}}{\boldsymbol{\partial q}} + \boldsymbol{\alpha} \frac{\boldsymbol{\partial c(q^*, h^*)}}{\boldsymbol{\partial q}} + \boldsymbol{t}$$

and from part B:

$$\boldsymbol{p} = \frac{\boldsymbol{\partial c(q^o, h^o)}}{\boldsymbol{\partial q}}\ , \boldsymbol{\phi'(h)} = \frac{\boldsymbol{\partial c(q^o, h^o)}}{\boldsymbol{\partial h}} \rightarrow \boldsymbol{\alpha \phi'(h^o) + t = 0} \rightarrow \boldsymbol{t = -\alpha \phi'(h^o)}$$

Thus if $t = -\alpha\phi'(h)$, then that the government taxes the firm on output level will restore efficiency.

11.C.1:

First let's find the optimal level of the public good. Assume that the cost of supplying q units of the public good is (q). Thus:

$$Optimal: \max\ V = \sum_{i=1}^{I} \phi_i(q) - c(q)$$

$$\frac{\partial V}{\partial q} = \sum_{i=1}^{I} \phi'_i(q) - c'^{(q)} \leq 0 \rightarrow \sum_{i=1}^{I} \boldsymbol{\phi'_i(q^o) \leq c'(q^o)}\ \ w.eq\ if\ q^o > 0$$

Now let's write the problem when we have subsidy for the public good. Also assume that a private firm produces public good and the price is p.

$$UMP\ for\ i:\ \max\ U = \phi\left(\sum x_i\right) + s_i x_i - p x_i$$

$$\frac{\partial U}{\partial x_i} = \phi'\left(\sum x_i\right) + s_i - p \leq 0 \rightarrow \boldsymbol{\phi'\left(\sum x_i\right) + s_i \leq p}\ \ w.eq\ if\ x_i > 0$$

$$PMP: \max\ pq - c(q)$$

$$\frac{\partial \pi}{\partial q} = p - c'(q) \leq 0 \rightarrow \boldsymbol{p \leq c'(q)}\ \ w.eq\ if\ q > 0$$

Now assume that $q > 0$, $x_i > 0$. Thus:

$$p = c'(q)\,, \sum_{i=1}^{I} \phi'_i(q^o) = c'(q^o)\,, \phi'_i\left(\sum x_i\right) + s_i = p \rightarrow$$

$$s_i = \sum_{i=1}^{I} \phi'_i\left(\sum x_i\right) - \phi'_i\left(\sum x_i\right) \rightarrow$$

For each individual j, if:

$$\boldsymbol{s_i = \sum_{i=1, i\neq j}^{I} \phi'_i\left(\sum x_i\right)}$$

Then we may reach optimal level through a subsidy on public good.

11.C.2:

Exactly like the previous problem. First let's solve for each individual:

$$UMP\ for\ i:\ \max U = \phi_i\left(\sum x_i\right) - p x_i$$

$$\frac{\partial U}{\partial x_i} = \phi'_i\left(\sum x_i\right) - p \leq 0 \rightarrow \boldsymbol{\phi'_i\left(\sum x_i\right) \leq p}\ \ w.eq\ if\ x_i > 0$$

And then for the firm:

$$PMP\text{: max } pq - c(q) + sq$$

$$\frac{\partial \pi}{\partial q} = p + s - c'(q) \leq 0 \rightarrow \boldsymbol{p + s \leq c'(q)} \;\; w.eq \; if \; q > 0$$

And we know that for the optimal level:

$$Optimal\text{: max } V = \sum_{i=1}^{I} \phi_i(q) - c(q)$$

$$\frac{\partial V}{\partial q} = \sum_{i=1}^{I} \phi'_i(q) - c'^{(q)} \leq 0 \rightarrow \boldsymbol{\sum_{i=1}^{I} \phi'_i(q^o) \leq c'(q^o)} \;\; w.eq \; if \; q^o > 0$$

Thus is we assume that $q > 0$, $x_i > 0$:

$$\boldsymbol{\phi'_i\left(\sum x_i\right) = p \, , p + s = c'(q) \;\; , \sum_{i=1}^{I} \phi'_i(q^o) = c'(q^o)}$$

The important point is $\phi_i'(\sum x_i) = p$ cannot be true.so assume that $k = argmax\left(\phi_i'(q^o)\right)$, and the price of the good is:

$$\boldsymbol{\phi'_k\left(\sum x_i\right) = p}$$

Then if we have:

$$s = \boldsymbol{\sum_{i \neq k}^{I} \phi'_i(q^o)}$$

Then we may reach optimal level through a subsidy for the firm.

11.D.2:

First let's solve for when we have tax of t_i for each individual:

$$UMP\text{: max } \phi(h_i, h_i + H_{-i}) + w_i - t_i h_i \quad , \quad H_{-i} = \sum_{j \neq i} h_j$$

$$\frac{\partial U}{\partial h_i} = \phi_i'(h_i, h_i + H_{-i}) + \frac{\partial \phi_i}{\partial H} - t_i \leq 0 \rightarrow \boldsymbol{\phi_i'(h_i, h_i + H_{-i}) + \frac{\partial \phi_i}{\partial H} \leq t_i} \; w.eq \; if \; h_i > 0$$

Now let's solve for the pareto optimal level:

$$\max \sum_{i=1}^{I} (\phi_i(h_i, h_i + H_{-i}) + w_i)$$

$$\frac{\partial V}{\partial h_i} = \phi_i'(h_i, h_i + H_{-i}) + \frac{\partial \phi_i}{\partial H} + \sum_{j \neq i}^{I} \frac{\partial \phi_j}{\partial H} \leq 0 \quad w.eq \;\; if \;\; h_i > 0$$

Now assume that $h_i > 0$. Then we have:

$$\phi_i'(h_i, h_i + H_{-i}) + \frac{\partial \phi_i}{\partial H} = t_i \,, \phi_i'(h_i, h_i + H_{-i}) + \frac{\partial \phi_i}{\partial H} + \sum_{j \neq i}^{I} \frac{\partial \phi_j}{\partial H} = 0$$

Thus if:

$$\boldsymbol{t_i} = \sum_{\boldsymbol{j \neq i}}^{\boldsymbol{I}} \frac{\boldsymbol{\partial \phi_j (h_j, H_{-j})}}{\boldsymbol{\partial H}}$$

Then we may restore the efficiency with a subsidy program.