Game Theory: the Basic Concepts

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Why Game Theory?

- Assumes the existence of many buyers and many sellers.
- Decision making is made independently and individually.
  - Decisions are not inter-dependent: Decision of one agent is neither influenced by another agent, nor does it influence that of another agent.
  - Decisions are not made in coalitions (together/jointly), no cooperative game.
- Independent and individual decision-making under perfect competition implies each decision-maker tries to do the best they can irrespective of what other decision-makers are doing.

What is Game Theory?

- Game theory is a formal way to analyze **interactions** among a group of **rational** agents who behave **strategically**.
  - **Interaction:** What one agent does directly affects at least one other agent
  - **Strategic:** Agents take into account that their actions influence the game
  - **Rational:** An agent chooses its best action (maximizes its expected utility)

What is Game Theory?

- Branch of applied mathematics and economics that studies strategic situations where there are several payoff maximizer players, whose actions can affect outcome of one another.
- The value of game theory lies in understanding the interactions between agents when potential conflict of interest exists.
- Game theory is mainly used in economics, political science, and marketing, market regulation, law legislation, family economics.
- The subject first addressed zero-sum games, today, game theory applies to a wide range of behavioral relations: non-cooperative and cooperative settings

Brief History of game theory

- 1738
  In a letter dated 13 November 1713 Francis Waldegrave provided the first known, minimax mixed strategy solution to a two-person game.
- 1838
  The French economist Antoine Augustine Cournot discussed a duopoly where the two duopolists set their output based on residual demand.
- 1928
  John von Neumann proved the minimax theorem. It states that every two-person zero-sum game with finitely many pure strategies for each player is determined.

Brief History of game theory

- Intellectual debate between Hungarian Mathematician John Von Neumann (1903-1957) and he Austrian economist Oskar Morgenstern (1902-1977): **Can utility be quantified?**
- 1944
  Theory of Games and Economic Behavior by John von Neumann and Oskar Morgenstern is published. As well as expounding two-person zero sum theory this book is the seminal work in areas of game theory such as the notion of a cooperative game, with transferable utility (TU), its coalitional form and its von Neumann-Morgenstern stable sets. It was also the account of axiomatic utility theory given here that led to its wide spread adoption within economics.

Brief History of game theory

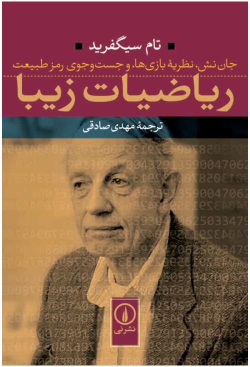
- 1950
  Contributions to the Theory of Games I, H. W. Kuhn and A. W. Tucker eds., published.
- 1950
  In the summer of 1950 Tucker was at Stanford University. He was working on a problem in his room when a graduate student of psychology knocked and asked what he was doing. The answer was short: game theory. ”Why don’t you explain to us in a seminar”? Tucker used his now famous example of two thieves who were put into separate cells and asked the same question by the judge. Tucker christened the phenomenon as **The Prisoners’ Dilemma**.

Brief History of game theory

- 1953
  Extensive form games allow the modeler to specify the exact order in which players have to make their decisions and to formulate the assumptions about the information possessed by the players in all stages of the game. In two papers, Extensive Games (1950) and Extensive Games and the Problem of Information (1953), H. W. Kuhn included the formulation of extensive form games which is currently used, and also some basic theorems pertaining to this class of games.

Brief History of game theory

- 1953
  In four papers between 1950 and 1953 John Nash (1928 - 2015) made seminal contributions to both non-cooperative game theory and to bargaining theory.



Brief History of game theory

- Nash arrived at Princeton in the Fall of 1948 to start a PhD. He came to Princeton with one of the shortest reference letters. His professor Richard Duffin (1909-1996) wrote just one sentence: ”This man is a genius.” Just eighteen months later, Nash submitted a 28 page doctoral dissertation.

- Every finite game must have at least one solution such that once reached, no player within the game will have an incentive to deviate from their chosen actions, given the actions of the other players in the game.

Brief History of game theory

- The intuition behind Nash's equilibrium: If all rational economic agents in a system are trying to do the best they can, assuming the others are doing the same, the system must be in equilibrium such that no single agent will want to unilaterally deviate from their position.

- A Nash equilibrium is not necessarily the best possible solution for all players in the game!

- If this theorem is true, how can one justify the invisible hand of Adam Smith?

- Perfect competition is a theoretical extreme. Like the ideal human body temperature of 98.4 degrees Fahrenheit (36.89C) it almost never exists. It is used as a benchmark to explain deviations from this 'perfect' world.

Cooperative and Non-cooperative Games

- There are two leading frameworks for analyzing games: cooperative and non-cooperative.

- **Cooperative games** are the one in which players are convinced to adopt a particular strategy through negotiations and agreements between players, Spouses' Game with bargaining and Prisoner's dilemma without any with a binding agreement.

- **Non-cooperative games** refer to the games in which the players decide on their own strategy to maximize their profit. Prisoner's dilemma without any agreement.

Cooperative and Non-cooperative Games

- As non-cooperative game theory is more general, cooperative games can be analyzed through the approach of non-cooperative game theory (the converse does not hold).

- Cooperative game theory assumes rationality, unlimited communication, and unlimited ability to make binding agreements.
- Non-cooperative game theory also assumes rationality.

- This course focuses on non-cooperative game theory.

see Osborn, a course in game theory.

Simultaneous Move Games and Sequential Move Games

- **Simultaneous games** are the one in which the move of two players (the strategy adopted by two players) is simultaneous.

- In simultaneous move, players do not have knowledge about the move of other players.

- **Sequential games** are the one in which players are aware about the moves of players who have already adopted a strategy.

Constant Sum, Zero Sum, and Non-Zero Sum Games

- **Constant sum game** is the one in which the sum of outcome of all the players remains constant even if the outcomes are different.
- **Zero sum game** is a type of constant sum game in which the sum of outcomes of all players is zero.
- **Non-zero sum game** are the games in which sum of the outcomes of all the players is not zero.

Perfect and imperfect information

- Perfect information refers to the fact that each player has the same information that would be available at the end of the game. This is, each player knows or can see other player's moves.

Example

A good example would be chess, where each player sees the other player's pieces on the board.

- Imperfect information appears when decisions have to be made simultaneously, and players need to balance all possible outcomes when making a decision.

Example

A good example of imperfect information games is a card game where each player's card are hidden from the rest of the players.

- The formal definition of these issues will be provided shortly.

What are the elements of a Game?

- From the noncooperative point of view, a game is a multi-person decision situation defined by its structure, which includes:
  - **Players:** Independent decision makers $i \in 1, 2, ..., I$
  - **Rules:** Which specify the order of players' decisions, their feasible decisions at each point they are called upon to make one, and the information they have at such points.
  - **Outcome:** How players' decisions jointly determine the physical outcome.
  - **Preferences:** players' preferences over outcomes.
- Preferences can be extended to handle shared uncertainty about how players' decisions determine the outcome as in decision theory, by assigning von Neumann-Morgenstern utilities, or payoffs, to outcomes and assuming that players maximize expected payoff.

Examples

- **Matching Pennies (version A).**
  - **Players:** There are two players, denoted 1 and 2.
  - **Rules:** Each player simultaneously puts a penny down, either heads up or tails up.
  - **Outcomes:** If the two pennies match, player 1 pays 1 dollar to player 2; otherwise, player 2 pays 1 dollar to player 1.

- **Matching Pennies (version B).**
  - **Players:** There are two players, denoted by 1 and 2.
  - **Rules:** Player 1 puts a penny down, either heads up or tails up. Then, Player 2 puts a penny down, either heads up or tails up, a really favorable game for player 2!
  - **Outcomes:** If the two pennies match, player 1 pays 1 dollar to player 2; otherwise, player 2 pays 1 dollar to player 1.

Examples

- **Matching Pennies (version C).**
  - **Players:** There are two players, denoted 1 and 2.
  - **Rules:** Player 1 puts a penny down, either heads up or tails up, without letting player 2 know his decision. Player 2 puts a penny down, either heads up or tails up.
  - **Outcomes:** If the two pennies match, player 1 pays 1 dollar to player 2; otherwise, player 2 pays 1 dollar to player 1.

- **Matching Pennies (version D).**
  - **Players:** There are two players, denoted 1 and 2.
  - **Rules:** Players flip a fair coin to decide who begins. The looser puts a penny down, either heads up or tails up. Then, the winner puts a penny down, either heads up or tails up.
  - **Outcomes:** If the two pennies match, player 1 pays 1 dollar to player 2; otherwise, player 2 pays 1 dollar to player 1.

The Extensive Form Representation of a Game

- One way to describe either kind of game is via the extensive form or game tree, which shows a game's sequence of decisions, information, outcomes, and payoffs.
- In game theory, the extensive form is away of describing a game using a game tree.

The Extensive Form Representation of a Game

- A version of Matching Pennies with sequential decisions, in which Player 1 moves first and player 2 observes 1's decision before 2 chooses his decision, **Version B**, a really favorable game for player 2.

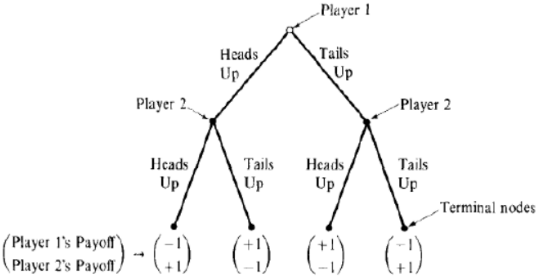
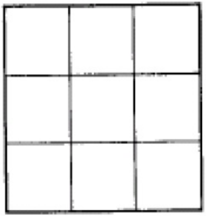


Figure 7.C.1
Extensive form for Matching Pennies Version B.

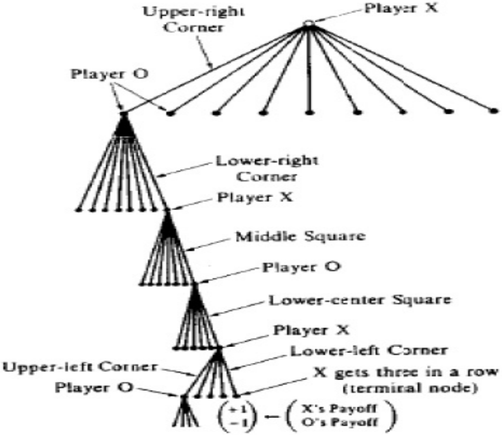
Elements of "Tic-Tac-Toe" Game and its Game board

- players (**player 1 and player 2**)
- Player 1 starts marking from upper-left corner by putting X, then player 2 puts her O in lower-right corner.
- The player who succeeds in placing three of their marks in a horizontal, vertical, or diagonal row is the winner. (**Rules of game**)
- Finally, the winner is paid 1 $ by loser. (**outcome of game**)
- **Payoff** is defined as utility of the outcome.



The Extensive Form Representation of a Game: "Tic-Tac-Toe"

- Extensive Form representation of a game includes all the paths may exist in a game.
- To save the space many parts are omitted.



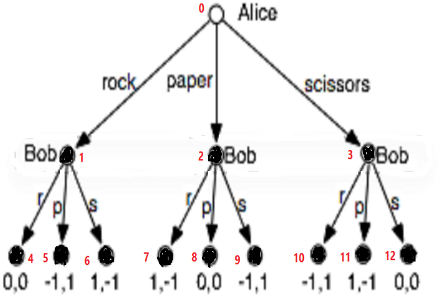
Dynamic Games

- In both Matching the Pennies **Version B** and **Tic-Tac-Toe**, when it is a player's turn to move, she is able to observe all her rival's previous moves, they are dynamic and **Games with perfect information**.

- See a game between man and computer!, I am going to choose X as my piece and make the first move.

Game with perfect information, Graphical representation

- A sequential version (far from reality!) for Rock, Paper and Scissors game with 2 players (Bob and Alice). How many nodes does it have? 13 nodes



the set of nodes is

$$\mathscr{X} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

Which includes the initial, decision and terminal nodes.

Formal representation of an extensive form game

1. A finite set of nodes $\mathscr{X} = \{0, 1, 2, ..., 12\}$ in the Rock, Paper and Scissors game.

2. A finite sets of actions (or strategies) [sometime our actions are infinite, e.g choosing a price level that maximizes our profit level, see Bertrand model of price competition] for player i, denoted by $\mathcal{A}_i = \{R, P, S\}$, where $i \in \{Alice, Bob\}$ and the possible **Actions Set** for the game, $\mathscr{A} = \mathcal{A}_1 \cup \mathcal{A}_2 = \{R, P, S\}$.

3. A finite set of players $\{1, 2, ..., I\}$ or $\{Alice, Bab\}$
   - In some games nature begins the first move (players flip a fair coin to find which palyer should do the first move), then set of players include zero $\{0, 1, 2, ..., I\}$

4. $x_0 \in \mathscr{X}$ is called initial node, and in our example node 0.

Formal representation of an extensive form game

5. The set of Terminal node is $T = \{4, 5, ..., 12\}$

6. The set of decision nodes is denoted by $D = \mathscr{X} \backslash T = \{0, 1, 2, 3\}$

7. the neighbors of given node are called **Predecessor** (the node lies behind of given node) and **Successor** (the node lies ahead of given node).

8. A function $p(x) : \mathscr{X} \to \{\mathscr{X} \cup \phi\}$ specifying a single immediate predecessor of each node x.
9. $p(x)$ is non-empty for all $x \in \mathscr{X}$, except the x_0. Why this is a function?

Example

$$p(8) = 2$$

Formal representation of an extensive form game

10. The immediate successor nodes of x are shown by $s(x)$ which is a correspondence. For the set T we have $s(x) = \{\}$, a null set for each x.

Example

$$s(2) = 8$$

11. A function $\alpha : \mathscr{X} \setminus \{x_0\} \to \mathscr{A}$ giving an action [for e.g R] that leads to any noninitial node x from its immediate predecessor $p(x)$.

Formal representation of an extensive form game

Example

Action function:

$$\alpha : \{1, 2, ..., 12\} \rightarrow \{R, P, S\}$$

Evaluate the action function in the non-initial node 2, $\alpha(2)$.
What immediate action leads to the node 2? answer clearly is P. This gives unique us action!.

Formal representation of an extensive form game

12. The set of choices (Actions for a player who is upon to move) available at decision node x is $c(x) = \{a \in \mathscr{A} : a = \alpha(x') \text{ for some } x' \in s(x)\}$.

Example

$c(2) = \{R\}$, such that $R = \alpha(8)$ for $8 \in s(2)$.

13. **Information set:** It matters very much what one knows when one makes a choice, namely, she does know that what has happened so far. this is the information partition. See the Node 2 for Bob. He knows that Alice has chosen to move *Paper*.
14. Collection of information sets for Bob is $H_{Bob} = \{\{1\}, \{2\}, \{3\}\}$ and that for Alice is $H_{Alice} = \{0\}$.

Formal representation of an extensive form game

15. Information collection for a game is defined as the collection of all information collections $\mathscr{H} = \{\{1\}, \{2\}, \{3\}, \{0\}\}$.

16. A function $\iota : \mathscr{H} \to \{0, 1, ..., I\}$ assigning each information set in $\mathscr{H}$ to the player who moves at the decision nodes.

Example

$$\iota(H_{Bob}) = Bob$$

17. **Payoff function**. A collection of payoff functions $\{u_i(.)\}_{i=1}^{I}$ assigning Bernoulli utilities to the players for each terminal node $\mathscr{A}$:

$$u_i(.) : T \to \mathbb{R}$$

Following our reference textbook we use T, but the exact definition for the domain of utility function is strategies space. We will define it shortly in the following sections.

Formal representation of an extensive form game

18. Formally, a game in extensive from is specified by the collection:

$$\Gamma_E = \{\mathscr{X}, \mathscr{A}, I, p(.), \alpha(.), \iota(.), \mathscr{H}, H(.), u\}$$

- we can add more arguments to this definition, for instance the probability of choosing an action in any information set, $\rho(.)$ which is function $\rho(.) : \mathscr{H}_0 \times \mathscr{A} \rightarrow [0, 1]$.
- Games with **complete information**. In the games with complete information, the value of payoffs are common knowledge.

Game with perfect information, formal definition

- In the games with imperfect information there exist non-singleton information sets, see the matching pennies **Version C** for player 2.

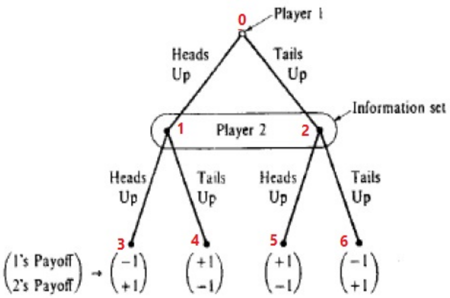
Definition

A game is one of perfect information if each information set contains a single decision node. Otherwise, it is a game of imperfect information.

- To summarize:
  - A sequential game is one of perfect information if each player knows every action of the players that moved before him at every point.
  - Technically, every information set contains exactly one node.
  - Intuitively, if it is my turn to move, I always know what every other player has done up to now.

Game with imperfect information: Example

- Recall the matching the pennies **Version C**, in which player 1 puts his coin down but covers it with his hand. How many nodes does it have? 7 nodes



information set of player is $H_2 = \{1, 2\}$ which is not a singleton.

- This game is similar to the matching the pennies **Version B** but, at first by flipping a coin, nature decides who will move first.

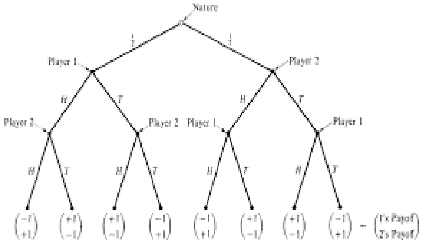


Figure 7.C.6 Extensive form for Matching Pennies Version D.

Strategies and the Normal Form Representation of a Game

- A strategy is a complete contingent plan for playing the game, which specifies a decision for each of a player's information sets in the game.
- A strategy is like a detailed manual of actions, not like a single decision or action.
- The player writes his own manual of actions. Then he will give you (a neutral referee) the manual and let you play out the game. You will tell him who won.

Example

Recall the matching pennies Version B

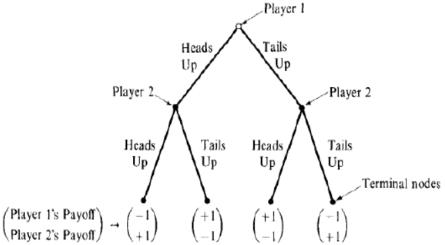


Figure 7.C.1
Extensive form for Matching Pennies Version B.

- What are the Pl's strategies? $S_1 = \{H, T\}$
- What are the P2's strategies? it depends!

Strategic Form Representation of a Game

1. Strategy 1: Play H if player 1 plays H; Play H if player 1 plays T
2. Strategy 2: Play H if player 1 plays H; Play T if player 1 plays T
3. Strategy 3: Play T if player 1 plays H; Play H if player 1 plays T
4. Strategy 4: Play T if player 1 plays H; Play T if player 1 plays T

Strategic form representation of matching pennies **Version B**

		Player 2's strategies			
		S1	S2	S3	S4
Player 1's strategies	S1	H H (-1,1)	H H (-1,1)	H T (+1,-1)	H T (+1,-1)
	S2	T H (+1,-1)	T T (-1,+1)	T H (+1,-1)	T T (-1,+1)

Strategic form representation of a Game

Example

The normal form game for the "Rock, Paper and Scissors". Set of strategies for Alice is, $S_{Alice} = \{R, P, S\}$ and similarly that for Bob is $S_{Bob} = \{R, P, S\}$, then the set of strategy profile is defined as:

$$S = S_{Alice} \times S_{Bob} = \{R, P, S\} \times \{R, P, S\}$$

$$= \{(RR), (RP), ..., (SS)\}$$

and the payoff is fun action $u_i : S \to \mathbb{R}$

Rock-Paper-Scissors over a Dollar

		Alice		
		R	P	S
Bob	R	$0, 0$	$-1, 1$	$1, -1$
	P	$1, -1$	$0, 0$	$-1, 1$
	S	$-1, 1$	$1, -1$	$0, 0$

Normal form representation of a Game

Example

Recall the rock, paper and scissors game. what are the Alice's strategies? and what are Bob's strategies? We represent the strategies by:
$S_{Alice} = \{R, P, S\}$ and $S_{Bob} = \{R, P, S\}$.

Definition

Normal form game: For a game with I players, the normal form representation Γ_N specifies for each player i a set of strategies S_i (with $s_i \in S_i$) and a payoff function $u_i(s_1, ..., s_I)$ giving the von Neumann-Morgenstern utility (ans Bernoulli utility function as well) levels associated with the (possibly random) outcome arising from strategies $(s_1, ..., s_I)$. Formally we write $\Gamma_N = [I, \{S_i\}, \{u_i(.)\}]$.

Randomized Choices in games: mixing actions in tennis game

- So far, we have assumed that the players are able to choose their actions with certainty.
- There are several situations that players cannot choose one specific action out of her actions set
- Imagine you and your friend are playing tennis, how do you shoot your tennis balls? Do you shoot all of them from **Right** to **Left** in the playground?
- Then your rival will react successfully and perhaps will dominate you.
- The same is true for your friend.
- So, you have to **mix** direction of your shootings, between *Down the Line* and *Cross Court*, and change your position as well.

Randomized Choices

Definition

A mixed strategy for player i is a function $\sigma_i : S_i \to [0, 1]$, which assigns a probability $\sigma_i(s_i) \geq 0$ to each pure strategy $s_i \in S_i$, satisfying $\sum_{s_i \in S_i} \sigma_i(s_i) = 1$.

Example

- Suppose that each player can randomize among all her strategies $S_1 = \{T, B\}$ and $S_2 = \{L, R\}$ by associating a probability distribution for elements of S_i.
- Strategy profile (T, L) is chosen by player one and two with probability of $p.q$, and so on for the other Strategy profiles.

			Player 2	
			q	$1-q$
			L	R
Player 1	p	T	pq	$p(1-q)$
	$1-p$	B	$(1-p)q$	$(1-p)(1-q)$

Randomized Choices

Example

- Expected (vN.M) utility function of the players one and two for mixed strategy profiles (p, q) are:

$$u_1(p, q) = pq.u_1(T, L) + p(1-q).u_1(T, R) +$$
$$(1-p)q.u_1(B, L) + (1-p)(1-q).u_1(B, R)$$

$$u_2(p, q) = pq.u_2(T, L) + p(1-q).u_2(T, R) +$$
$$(1-p)q.u_2(B, L) + (1-p)(1-q).u_2(B, R)$$

Randomized Choices

- **Normal form for games with mixed strategies:** Suppose that player i has M strategies in $S_i = \{s_{1i}, ..., s_{Mi}\}$, then her mixed strategies can be associated with infinite sets of lotteries: $\Delta(S_i) = \{(\sigma_{1i}, ..., \sigma_{Mi}) \in \mathbb{R}^M : \sigma_{mi} \geq 0$ for all $m = 1, ..., M$ and $\sum_{m=1}^{M} \sigma_{mi} = 1\}$.

- We denote the normal form game in the presence of uncertainty as $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(.)\}]$ in which the strategy sets are extended to include both **pure** and **mixed** strategies, why?

- Example: $\sigma_{11} = p$ and $\sigma_{12} = q$

Expected utility function: a general form for mixed strategies

- Recall that strategy space for a game is $S = S_1 \times .... \times S_I = \{(T, L), (T, R), (B, L), (B, R)\}$, where its typical element is $s = (s_1, ..., s_I)$, e.g. $s = (T, L)$.
- The strategy profile is realized with probability of $(\sigma_1 \times ... \times \sigma_I)$.
- Then, the vN.M utility function is represented by:

$$u_i(\sigma) = \sum_{s \in S} [\sigma_1(s_1) \times \sigma_2(s_2) \times ... \times \sigma_I(s_I)].u_i(s)$$

- Example: $\sigma_1(T) = p$ and $\sigma_2(L) = q$

Mixed strategies: A numerical example

- Recall that Matching pennies **version A** with the following strategic form:

			Player 2	
			0.5	0.5
			H	T
Player 1	p	H	1,-1	-1, 1
	1-p	T	-1,1	1,-1

- Assume that both of the players randomize their strategies with $p = 0.5$ and $q = 0.5$, Calculate the expected utilities.

- Imagine that player 1 always plays H, but player 2 randomizes between H and T, or in a technical term $p = 1$ and $q = 0.5$. Find the expected utilities for players 1 and 2.

- was the player 1 able to improve her payoff by this deviation?