DYNAMIC GAMES: SPNE

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OUTLINE

INTRODUCTION

- So far, we have studied games with Nash (Pure or Mixed) Equilibria and games with Bayesian perfect Equilibria

		Timing	
		Simultaneous	Sequential
Information	Complete	**Nash**	**Sub-game Perfect Nash Equilibrium**
	Incomplete	**Bayesian Nash**	**?**

INTRODUCTION

- This part intends to analyze dynamic games that have not only complete but also perfect information.
  - At each move in the game the player with the move knows the full history of the play of the game thus far.
- We now turn to study of sequential or dynamic games, i.e. games where players are not moving simultaneously, but rather in a sequence over time.

- We, again, restrict attention to games with complete information
  - i.e., games in which the players' payoff functions are common knowledge

INTRODUCTION

SUMMARY

The key features of a dynamic game of complete and perfect information are that:

1. the moves occur in sequence,
2. all previous moves are observed before the next move is chosen,
3. the players' payoffs from each feasible combination of moves are common knowledge.

- Central issue in all dynamic games is credibility.

- The underlying theme will be to refine the set of Nash equilibria in these games.
  - The problem is that certain Nash equilibria in dynamic games can be very implausible predictions.

INTRODUCTION

- The Nash equilibrium concept does not suffice to rule out non-credible strategies

- We will introduce a stronger solution concept, known as ***Subgame Perfect Nash Equilibrium***, that helps to do so.

- The central idea underlying this concept is the ***Principle of Sequential Rationality***.

DEFINITION (PRINCIPLE OF SEQUENTIAL RATIONALITY)

Equilibrium strategies should specify ***optimal behaviour*** *from any point in the game onward.*

INTRODUCTION

- The principle is intimately related to the procedure of ***Backward Induction***.
- The concept of subgame perfection is not strong enough to fully capture the idea of ***Sequential Rationality*** in games of imperfect information
- We then introduce the notion of of a ***Weak Perfect Bayesian equilibrium*** to find equilibrium strategies with ***beliefs***.
- Player's Beliefs explicitly shows what has occurred prior to her move as a means of testing the sequential rationality of the player's strategy.
- Why we use the adjective ***Weak***?

EXAMPLE (NON-CREDIBLE THREAT)

Consider the following two-move game.

- Suppose player 2 threatens to explode the grenade unless player 1 pays the $1,000.
- If player 1 believes the threat, then player l's best response is to pay the $1,000.
- Noncredible threat:
  - But player 1 should not believe the threat, because it is Non-credible.
  - if player 2 were given the opportunity to carry out the threat, he would choose not to carry it out.
  - Thus, player 1 should pay player 2 nothing.

EXAMPLE (IMPLAUSIBILITY OF SOME NASH EQUILIBRIA)

- Consider the following game, given in both normal-form and extensive-form.

P2

		Fight	Accommodate
	Out	1, 2	1, 2
P1	Enter	0, 0	2, 1

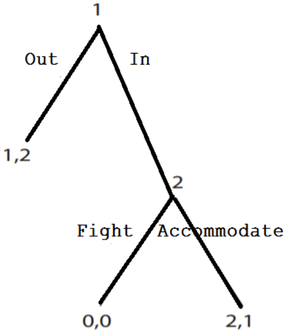
- This game has two Nash equilibria: (**Out**, **Fight**) and (**Enter(In)**, **Accommodate**).
- If we think of the players as selecting plans before the game starts, then the NE profile (**Out**, **Fight**) makes some sense (a potential situation), it is a rational choice ex ante.

THE IDEA OF NON-CREDIBLE THREADS: EXAMPLE

- Note that in that game, some of the **Nash** equilibria seem distinctly less intuitive that others.
- In the (**Out**, **Fight**) equilibrium, it is the threat of **Fight** that keeps Firm 2 from entering. However, if Firm 2 were to enter, is it reasonable to think that Firm 1 will actually **fight**?
- At this point, it is not in Firm 1's interest to **Fight**, since it does better by **Accommodating**.
- In fact, she would do everything in her power to convince player 1 that she will play **Out**, so that he plays **Out**.

THE IDEA OF NON-CREDIBLE THREADS: EXAMPLE

- While player 2 might "threaten" to play **Fight** before the game starts, if the game proceeds and it is actually her turn to move, then **Out** is not rational.



- The threat is not credible, and player 1 should realize this and play **Enter(In)**.

- Thus, **Fight** might be a reasonable plan for player 2 if she thinks player 1 will play **Out**, but it is not rational to carry out the plan after player 1 plays **Enter(In)**.

- It is not **sequentially rational** for player 2 to play **Fight**.

DEFINITION (SEQUENTIALLY RATIONAL STRATEGY)

A player's strategy exhibits sequential rationality if it maximizes his or her expected payoff, conditional on every information set at which he or she has the move. That is, player *i*'s strategy should specify an optimal action at each of player *i*'s information sets, even those that player *i* does not believe will be reached in the game.

DYNAMIC GAMES OF COMPLETE AND PERFECT INFORMATION

- There are games that have multiple Nash equilibria, some of which are unrealistic.
- In the case of dynamic games, unrealistic Nash equilibria might be eliminated by applying ***backward induction***, which assumes that future play will be rational.
- It therefore eliminates non-credible threats because such threats would be irrational to carry out if a player was ever called upon to do so.

DYNAMIC GAMES OF COMPLETE AND PERFECT INFORMATION

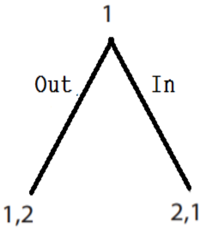
- For games of perfect information (all singleton information sets), backward induction is the process of "looking ahead and working backwards" to solve a game based on common knowledge of sequential rationality:

  ❶ Start at each node that is an immediate predecessor of a terminal node, find the optimal action for the player who moves at that node, and change that node into a terminal node with the payoffs from the optimal action.

  ❷ Apply step 1 to smaller and smaller games until we can assign payoffs to the initial node of the game.

- Looking ahead to player 2's decision node, her optimal choice is **Accommodate**, so we can convert her decision node into a terminal node with payoffs (2,1).
- In the smaller game, player 1 can either choose **Out** and reach the terminal node with payoffs (1,2), or **Enter(In)** and reach the terminal node with payoffs (2,1).
- Player 1's optimal choice is **Enter(In)**. Backward induction leads to the strategy profile (**Enter(In)**, **Accommodate**), with payoffs (2,1).

DYNAMIC GAMES OF COMPLETE AND PERFECT INFORMATION



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DYNAMIC GAMES OF COMPLETE AND PERFECT INFORMATION

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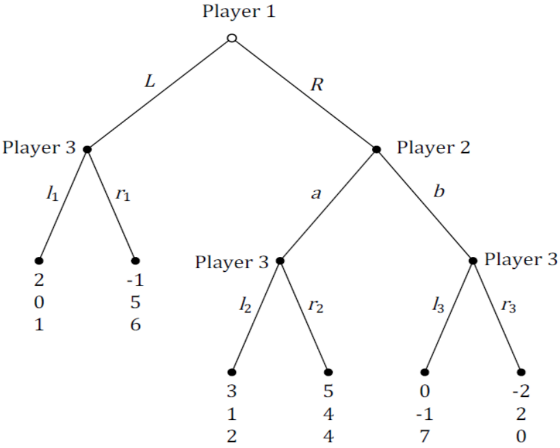
- Backward induction leads to the strategy profile (**Enter(In)**, **Accommodate**), with payoffs (2,1).

DYNAMIC GAMES OF COMPLETE AND PERFECT INFORMATION, MWG 9.B.3

- Consider the three-player finite game of perfect information



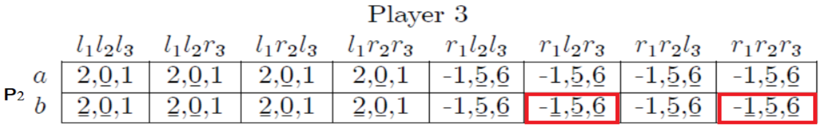
Extensive Form Game

DYNAMIC GAMES OF COMPLETE AND PERFECT INFORMATION, MWG 9.B.3

- Player 1's action set $S_1 = \{L, R\}$
- Player 2's action set $S_2 = \{a, b\}$
- Player 3's action sets $S_{31} = \{l_1, r_1\}$, $S_{32} = \{l_2, r_2\}$ and $S_{33} = \{l_3, r_3\}$
- Player 3's entire strategy set $S_3 = \{l_1, r_1\} \times \{l_2, r_2\} \times \{l_3, r_3\}$
- Then $S_3 = \{(l_1 l_2 l_3), (l_1 l_2 r_3), (l_1 r_2 l_3), (l_1 r_2 r_3), (r_1 l_2 l_3), (r_1 l_2 r_3), (r_1 r_2 l_3), (r_1 r_2 r_3)\}$

DYNAMIC GAMES OF COMPLETE AND PERFECT INFORMATION, MWG 9.B.3

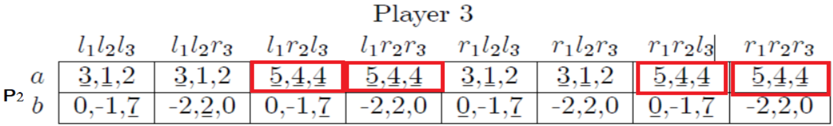
- Normal form game of perfect information
- From this normal form representation, we can identify six pure strategy Nash Equilibria:



Player 3

		$l_1l_2l_3$	$l_1l_2r_3$	$l_1r_2l_3$	$l_1r_2r_3$	$r_1l_2l_3$	$r_1l_2r_3$	$r_1r_2l_3$	$r_1r_2r_3$
P2	a	2,0,1	2,0,1	2,0,1	2,0,1	-1,5,6	-1,5,6	-1,5,6	-1,5,6
	b	2,0,1	2,0,1	2,0,1	2,0,1	-1,5,6	-1,5,6	-1,5,6	-1,5,6

Player 1: L



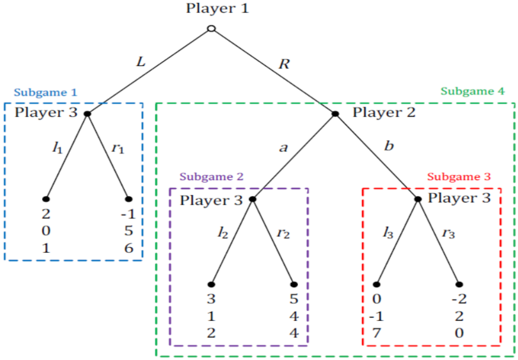
Player 3

		$l_1l_2l_3$	$l_1l_2r_3$	$l_1r_2l_3$	$l_1r_2r_3$	$r_1l_2l_3$	$r_1l_2r_3$	$r_1r_2l_3$	$r_1r_2r_3$
P2	a	3,1,2	3,1,2	5,4,4	5,4,4	3,1,2	3,1,2	5,4,4	5,4,4
	b	0,-1,7	-2,2,0	0,-1,7	-2,2,0	0,-1,7	-2,2,0	0,-1,7	-2,2,0

Player 1: R

- To find the SPNE, we must perform backwards induction on our extensive form game.

- Proper Subgames of the Extensive Form
- Starting with subgames 1, 2 and 3, we can evaluate player 3's decisions at each subgame



Proper Subgames of the Extensive Form

- It is clear that in subgame 1, player 3 will choose strategy r_1 since his payoff of 6 from selecting r_1 is greater than his payoff of 1 from selecting l_1.

EXAMPLE (COLD WAR BETWEEN SOVIET UNION AND UNITED STATES)

- An international crisis in October 1962, the closest approach to nuclear war at any time between the US and the Soviet Union.

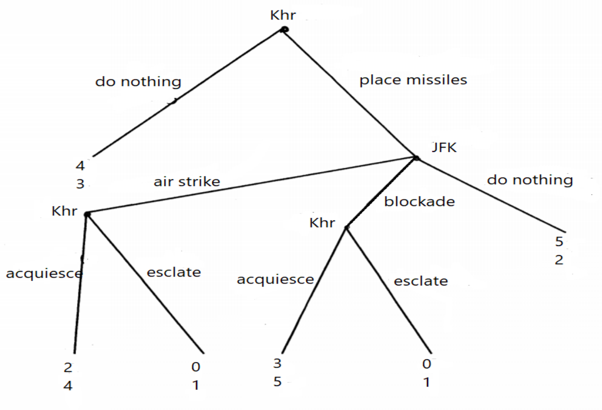
- When the US discovered Soviet nuclear missiles on Cuba, President John F. Kennedy demanded their removal and announced a naval blockade of the island; the Soviet leader Nikita Sergeyevich Khrushchev acceded to the US demands a week later.

- When the US found out, President Kennedy discussed the options (i) do nothing, (ii) air strike on the missiles, (iii) a naval blockade of Cuba.

Dynamic Games of Complete and perfect Information

Example (Cold War between Soviet Union and United States)

- If the missiles are in place, JFK must decide on (i) nothing, (ii) air strike, or (iii) blockade.



A MORE COMPLICATED EXAMPLE: CUBAN MISSILE CRISIS.

Example (Cold War between Soviet Union and United States, Cont.)

- If JFK decides on air strike or blockade, Khrushchev must decide whether to acquiesce or escalate.
- JFK would optimally choose blockade, leading to payoff (3,5).
- Khrushchev's optimal choice is status quo (4,3), since he receives a higher payoff than placing the missiles (3,5).

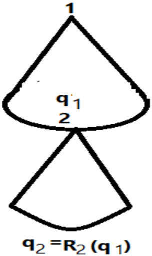
DYNAMIC GAMES OF COMPLETE AND PERFECT INFORMATION

- Common feature of the games just studies are dynamic in moves, and with complete and perfect information:
  - first player 1 moves, then player 2 observes player l's move, and takes its action. Finally, player 1 moves and the game ends.

Example

Stackelberg model of duopoly game as a Motivating example for backwards induction

- Stackelberg's (1934) model of duopoly belongs to the Dynamic games class.
- Two payoff maximizer firms with $u_1(s_1, s_2)$ and $u_2(s_1, s_2)$
- Firm **one** is **leader** in the market and Firm **two** is **follower**



EXAMPLE

Stackelberg model of duopoly game, cont.

- Both the firms are aware of the payoff structures and rule of the game.
    - The leader (Firm 1) will take the first action and then, after observing the action by firm 2, Firm 2 will decide which action maximizes its payoff
- Firm 1 is aware of the way that firm 2 makes its decision. Firm 1 will take that information in its consideration in stage 1 of the game.
- Firm 2's decision depends on s_1, i.e. given the action s_1 previously chosen by player 1:

$$max_{s_2 \in S_2} u_2(s_1, s_2)$$

EXAMPLE

Stackelberg model of duopoly game, cont.

- Assume that for each $s_1 \in S_1$, player 2's optimization problem has a unique solution such that $s_2 = R_2(s_1)$, the **player 2's reaction to player 1's action**.
- Since player 1 can solve 2's problem as well as 2 can, player 1 should anticipate player 2' s reaction to each action that 1 might take, so 1's problem at the first stage amounts to:

$$max_{s_1 \in s_1} u_1(s_1, s_2 = R_2(s_1))$$

- Assume that this optimization problem for player 1 also has a unique solution, denoted by s_1^*.
- The $(s_1^*, R_2(s_1^*))$ is called ***backwards− induction*** outcome of this game.

EXAMPLE (STACKELBERG DUOPOLY)

- Firm 1 leader and Firm 2 follower
- The payoffs are profit function π_i

$$\pi_i(q_i, q_j)) = [p(Q) - c]q_i$$

where $Q = q_i + q_j$ and $p = a - Q$

- To solve for the backwards-induction outcome of this game, we first find firm 2's reaction to an arbitrary quantity by firm 1, which gives us $q_2 = R(q_1)$.

$$max_{q_2} \pi_2(q_1, q_2) = max_{q_2}[a - q_1 - q_2 - c]q_2$$

which yields $R(q_1) = \frac{a-q_1-c}{2}$

- the payoffs and the reaction function, by assumption, are common knowledge.

EXAMPLE (STACKELBERG DUOPOLY, CONT.)

- Thus, firm l's problem in the first stage of the game amounts to:

$$max_{q_1}\pi_1(q_1, R(q_1)) = max_{q_1}[a - q_1 - R(q_1) - c]q_1$$

- which yields $q_1^* = \frac{a-c}{2}$ and $R(q_1^*) = \frac{a-c}{4}$
- Compare them with the equilibrium strategies for Cournot game in which each firm produces $\frac{a-c}{3}$.
- Informational advantages of the Leader Firm $[q_2 = R(q_1)]$ allows it to have higher share in the market

SUMMARY: BACKWARD INDUCTION

- ***In Summary***. *We can apply this logic to any extensive game in the following way:*
  *start at the **"end"** of the game tree, and work **"back"** up to initial node of the tree by solving for optimal behavior at each node.*
- This procedure is known as backward induction. In the class of finite games with perfect information (finite number of nodes and singleton information sets), this is a powerful procedure.
- The concept of backward induction corresponds to the assumption that it is common knowledge that each player will act rationally at each future node where he moves even if his rationality would imply that such a node will not be reached.

SUMMARY: BACKWARD INDUCTION

THEOREM

Zermelo. *Every finite game of perfect information has a pure strategy Nash equilibrium that can be derived through backward induction. Moreover, if no player has the same payoffs at any two terminal nodes, then backward induction results in a unique Nash equilibrium.*

SUMMARY: BACKWARD INDUCTION

- Every finite game of perfect information has a pure strategy Nash equilibrium.
- Zermelo's Theorem says that backward induction can be powerful in various finite games.
- For example it implies that even a game as complicated as chess is solvable through backward induction, if one is able to design its extensive form for all of contingent plans.
- In this sense, chess is "solvable" although, no-one knows what the solution is!

SUBGAME PERFECTION

- In extensive-form games with imperfect information backward induction can be problematic, because a player's optimal action depends on which node she is at in her information set.
- Still, sequential rationality can be captured by the concept of subgame perfection.
- What do we mean by a subgame?

DEFINITION

Given an extensive-form game tree, a node x initiates a **subgame** if neither x nor any of its successors are in an information set containing nodes that are not successors of x. The tree defined by x and its successors is called a **subgame**.

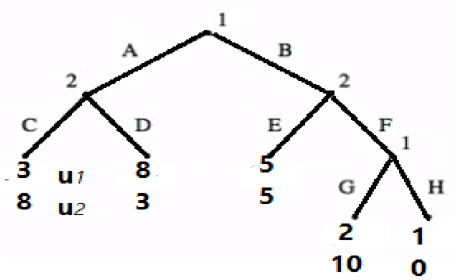
SUBGAME PERFECTION

- Notice that:
  1. Any game is a subgame of itself. Subgames other than the original game itself are called proper subgames.
  2. For games of perfect information, every node other than a terminal node defines a subgame.
  3. Any subgame is a game in its own right, satisfying all of our rules for game trees.

SUBGAME-PERFECT NASH EQUILIBRIA

EXAMPLE (A GAME WITH SEVRAL SUBGAMES)

- Here is a game with 4 subgames.
  1. Find the Strategic form representation
  2. Find all Nash equilibria
  3. Find all Subgame-Perfect Nash Equilibria



SUBGAME-PERFECT NASH EQUILIBRIA

EXAMPLE (A GAME WITH SEVRAL SUBGAMES)

- Sets of player one's actions at each information set is: $A_1 = \{A, B\}$ and $A_2 = \{G, H\}$
- So, her strategy set is derived as the Cartesian product of the sets:

$$S_1 = \{A, B\} \times \{G, H\} = \{AG, AH, BG, BH\}$$

- In the same manner we can find that for player two

$$S_2 = \{C, D\} \times \{E, F\} = \{CE, CF, DE, DF\}$$

- Then, the set of strategy profiles are:

$$S = S_1 \times S_2 = \{(AG, CE), (AH, BE), ...\}$$

SUBGAME-PERFECT NASH EQUILIBRIA

EXAMPLE (A GAME WITH SEVRAL SUBGAMES)

- the Strategic form representation of the game
- Two Nash Equilibria, Which on is credible?

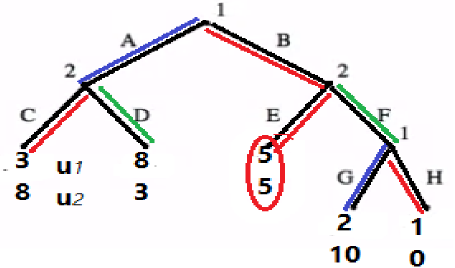
		Player 2			
		CE	CF	DE	DF
Player 1	AG	3, 8	**3**, **8**	8, 3	8, 3
	AH	3, 8	**3**, **8**	8, 3	8, 3
	BG	5, 5	2, 10	5, 5	2, 10
	BH	**5**, **5**	1, 0	5, 5	1, 0

- One of the Nash equilibrium strategies is $\sigma = (BH, CE)$, is that credible?
- Let's take a look on the extensive form game in more details

Subgame-Perfect Nash Equilibria

Example (A game with sevral subgames)

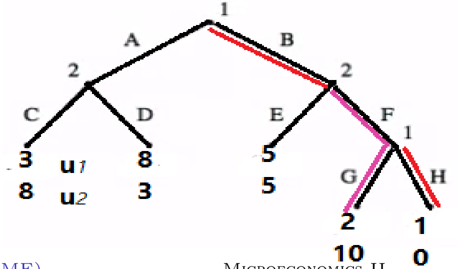
- The pure strategy $\sigma = (BH, CE)$ is represented by red lines in the figure
- Can player two profitably deviate from her current choice? What can she do? see the green lines
- Can player two can profitably deviate from her current choice? what could he do? follow the blue lines



SUBGAME-PERFECT NASH EQUILIBRIA

EXAMPLE (A GAME WITH SEVRAL SUBGAMES)

- So, neither of the players can profitably deviate from their current strategy.
- Again assume that player 1, is going to choose **BH** and let's focus on this subgame.
- Why wouldn't player 1 actually do **H**? Because a **G** dominates it, an the Nash Equilibrium is **G**
- Why wouldn't player 2 actually do **F** rather than **E**? there is non-credible threat **H** by player 1.

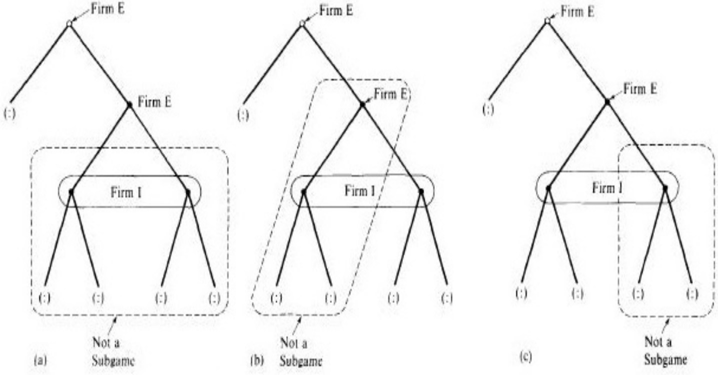


Subgame-Perfect Nash Equilibria

- The examples show that, in finite games with perfect information every decision initiates a sub-game.

Example (A game with imperfect information)

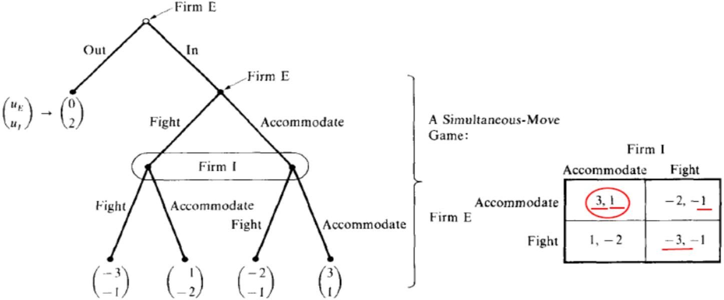
- Parts of a game which is not a sub-game



SUBGAME-PERFECT NASH EQUILIBRIA

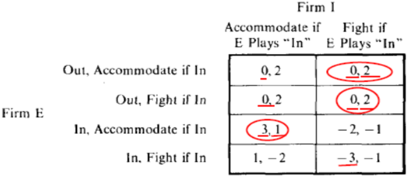
EXAMPLE (PRINCIPLE OF SEQUENTIAL RATINALITY FOR GAMES OF IMPERFECT INFORMATION)

- Consider the following two person game played by an Entrant **E** and an Incumbent **I**.



A Simultaneous-Move Game:

		Firm I	
		Accommodate	Fight
Firm E	Accommodate	3, 1	−2, −1
	Fight	1, −2	−3, −1



		Firm I	
		Accommodate if E Plays "In"	Fight if E Plays "In"
Firm E	Out, Accommodate if In	0, 2	0, 2
	Out, Fight if In	0, 2	0, 2
	In, Accommodate if In	3, 1	−2, −1
	In, Fight if In	1, −2	−3, −1

SUBGAME-PERFECT NASH EQUILIBRIA

EXAMPLE (PRINCIPLE OF SEQUENTIAL RATINALITY FOR GAMES OF IMPERFECT INFORMATION)

- Sets of player **E**'s actions: $S_{E1} = \{Out, In\}$, $S_{E2} = \{Fight, Accommodate\}$, accordingly her strategies set is: $S_E = S_{E1} \times S_{E2}$
- Set of player **I**'s actions: $S_{12} = \{Fight, Accommodate\}$
- In the game there are three pure three pure strategy Nash Equilibria (σ_E, σ_I)
  - ((*Out, Acc if In*), (*fight if E plays "In"*))
  - ((*Out, Fight if In*), (*fight if E plays "In"*))
  - ((*In, Acc if In*), (*Acc if E plays "In"*))
- (*Acc, Acc*) is the sole Nash Equilibrium in post entry game.
- payoff from **entry** for Player **E**, 3 is greater than 0 staying **out**

SUBGAME PERFECT NASH EQUILIBRIUM

DEFINITION

subgame perfect Nash equilibrium: A strategy profile for an extensive-form game is a subgame perfect Nash equilibrium (SPNE) if it specifies a Nash equilibrium ***in each of its subgames***.

- Every SPNE must also be a NE, actually by this way we are refining the NEs.
- For finite games of perfect information, any backward induction solution is a SPNE and vice-versa.
- The advantage of SPNE is that it can be applied to games of imperfect information too.

SUBGAME-PERFECT WITH MULTIPLE NASH EQUILIBRIA

EXAMPLE

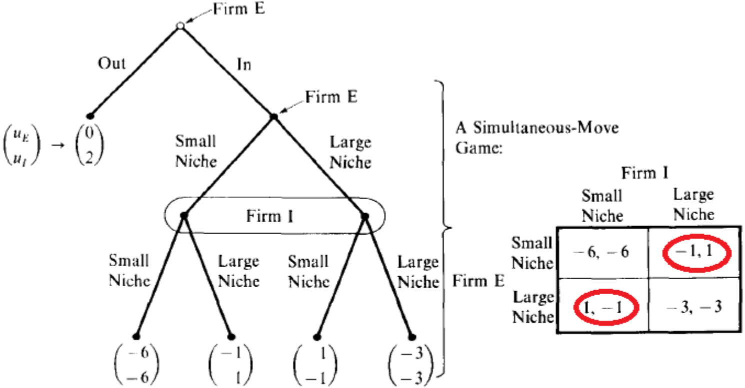
- Sets of player **E**'s actions at each information set is: $E_1 = \{Out, In\}$ and $E_2 = \{Small\ \ Niche, Large\ \ Niche\}$
- Set of player **I**'s actions at its information set is: $I_1 = \{Small\ \ Niche, Large\ \ Niche\}$
- Then, the set of strategy profiles of the Game is derived by:

$$S = E_1 \times E_2 \times I_1 = \{(Out, Small\ \ \ Niche, Small\ \ Niche),$$

$$(Out, Small\ \ Niche, Large\ \ Niche), ...\}$$

SUBGAME-PERFECT WITH MULTIPLE NASH EQUILIBRIA

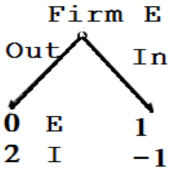
- Lets consider the example of Predation with Niches: Firm **E**, (the potential Entrant) first chooses to enter or not. If it enters, then the two firms simultaneously choose a **Niche** of the market to compete.



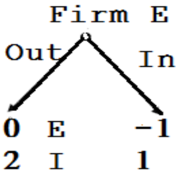
		Firm I	
		Small Niche	Large Niche
Firm E	Small Niche	−6, −6	−1, 1
	Large Niche	1, −1	−3, −3

Subgame-Perfect with multiple Nash Equilibria

- We have two pure Nash equilibria in the post Entry subgame.



(σ_E, σ_I) =((in, large niche if in), (small niche if firm E palys "in")



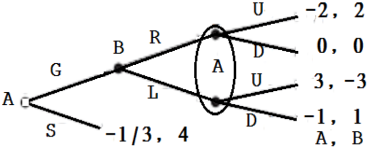
(σ_E, σ_I) =((out, small niche if in), (large niche if firm E plays "in")

-

Subgame-Perfect with multiple Nash Equilibria

Example (Multiple SPE)

- Multiple SPE may exist even if each payoff is unique.
- Rather than working from the very bottom decision node, we work from the last decision node in the game with a unique history. We call this a sub-game.



$$A_1 = \{G, S\}$$
$$A_2 = \{U, D\}$$
$$A_1 \times A_2 = \{GU, GD, SU, SD\}$$

Player B

Player A

	L	R
U	3, -3	-2, 2
D	-1, 1	0, 0

Table: No Pure Nash Strategy for this sub-game

Player B

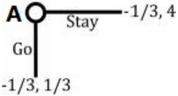
Player A

	L	R
GU	3, -3	-2, 2
GD	-1, 1	0, 0
SU	-1/3, 4	-1/3, 4
SD	-1/3, 4	-1/3, 4

Table: Four Pure Nash Strategies for the whole game

SUBGAME-PERFECT WITH MULTIPLE NASH EQUILIBRIA

- Multiple SPE may exist even if each payoff is unique.
- With that information in hand, we erase the subgame and replace it with those from expected payoffs ($p_U^* = \frac{1}{6}$, $q_L^* = \frac{1}{3}$).
- Because, from $-3p_U + (1 - p_U) = 2p_U$ one can get $p_U^* = \frac{1}{6}$, and with the same manner, $3q_L - 2(1 - q_L) = -q_L$ and $q_L^* = \frac{1}{3}$. Then $3\frac{1}{3} - 2\frac{2}{3} = -\frac{1}{3}$



- He earns -1/3 regardless of whether he chooses stay or go.
- He can select either as a pure strategy or play any mixture between the two, the third case in the following slide.

SUBGAME-PERFECT WITH MULTIPLE NASH EQUILIBRIA

- Different types of **SPE**.
  - In the **first**, player 1 **Goes** as a pure strategy, giving us an SPE of $((Go, p_U^* = \frac{1}{6}), q_L^* = \frac{1}{3})$.
  - In the **second**, player 1 **Stays** as a pure strategy, $((Stay, p_U^* = \frac{1}{6}), q_L^* = \frac{1}{3})$.
  - Finally, $((p_G, p_U^* = \frac{1}{6}), q_L^* = \frac{1}{3})$ represents the cases where player 1 mixes between **Stay** and **Go**, where p_G equals any number between 0 and 1, not including 0 and 1.

CRITICISMS OF SUBGAME PERFECTION

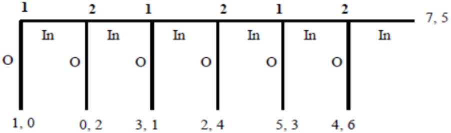
- We motivated Subgame Perfection as an attempt to eliminate equilibria that involved incredible threats.
- As we go on to consider applications, we will use SPE regularly as a solution concept.
- Before we do this, however, it is worth pausing momentarily to ask whether SPE might be fanatical in eliminating equilibria.

CRITICISMS OF SUBGAME PERFECTION

EXAMPLE (THE CENTIPEDE GAME)

- In games with many stages, backward induction greatly stresses the assumption of rationality (and common knowledge of rationality).

- The unique SPE is for Player 1 to start by moving Out

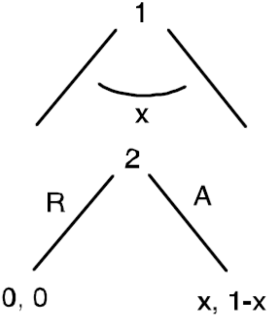


- The subgame perfect outcome is unfortunate; many outcomes have better payoffs for both players than what occurs in equilibrium.

CRITICISMS OF SUBGAME PERFECTION

EXAMPLE (ULTIMATUM GAME: A SIMPLE VERSION)

- Two players have a continuous dollar (or a Cake) to divide.
- Player 1 proposes to divide the dollar at $x \in [0, 1]$, where he will keep x and player 2 will receive $1 - x$.
- Player 2 can choose to either **Accept** this division of the dollar, or **Reject** it, in which case each player receives 0.



CRITICISMS OF SUBGAME PERFECTION

EXAMPLE (ULTIMATUM GAME: A SIMPLE VERSION)

- Any division $(p, 1-p)$ of the dollar can be sustained as a Nash equilibrium:

$$1 : x = p \tag{1}$$

$$2 : \begin{cases} R & \text{if } 1-x < 1-p \\ A & \text{if } 1-x \geq 1-p \end{cases} \tag{2}$$

- Notice that 2's strategy specifies how he responds to any offer, not just the one that 1 actually makes in equilibrium.

CRITICISMS OF SUBGAME PERFECTION

EXAMPLE (ULTIMATUM GAME:)

- We might interpret this equilibrium as "*2 demands at least* $1-p$*, and 1 offers 2 the minimal amount that 2 will accept.*"

- In the case of $p < 1$, however, player 2's strategy involves a not believable threat to reject a positive amount $1-p$ in favor of 0. **Not rational behavior.**

- Remember from the definition of Nash equilibrium that $p = 0$ or 1 is a possible Nash equilibrium as well.

$$1 : x = p \tag{3}$$

$$2 : A \tag{4}$$

- However, experimental trials show that (0.5, 0.5) strategy is a focal point, in terms of fairness

CRITICISMS OF SUBGAME PERFECTION

- In practice people do not seem to play the game this way.
- The centipede game is frequently subject of laboratory experiments.
- Game theorists have a variety of explanations for the discrepancy between subgame perfect play and play in practice.
- As with everything else in game theory, backward induction is only as good as its assumptions.

REPEATED GAMES

EXAMPLE

- Many interactions in the real world have an ongoing structure
  - Firms compete over prices or capacities repeatedly
- In such situations players consider their long-term payoffs in addition to short-term gains
- This might lead them to behave differently from how they would in one-shot interactions
- Consider the following pricing game in the DRAM chip industry

		Player B	
		High	Lowh
Player A	High	2, 2	0, 3
	Low	3, 0	1, 1

TABLE: No Pure Nash Strategy for this subgame

DYNAMIC RIVALRY

EXAMPLE

- If a firm cuts its price today to steal business, rivals may retaliate in the future, nullifying the ”benefits” of the original price cut
- In some concentrated industries prices are maintained at high levels
  - U.S. steel industry until late 1960s
  - U.S. cigarette industry until early 1990s
- In other similarly concentrated industries there is intense price competition
  - Costa Rican cigarette industry in early 1990s
  - U.S. airline industry in 1992
- When and how can firms sustain collusion?
- They could formally collude by discussing and jointly making their pricing decisions
  - Illegal in most countries and subject to severe penalties

IMPLICIT COLLUSION

EXAMPLE

- Could firms collude without explicitly fixing prices?
- There must be some reward/punishment mechanism to keep firms in line
- Repeated interaction provides the opportunity to implement such mechanisms
- For example Tit-for-Tat Pricing: mimic your rival's last period price
- A firm that contemplates undercutting its rivals faces a trade-off
  - short-term increase in profits
  - long-term decrease in profits if rivals retaliate by lowering their prices

IMPLICIT COLLUSION

EXAMPLE

- Depending upon which of these forces is dominant collusion could be sustained
- What determines the sustainability of implicit collusion?
- Repeated games is a model to study these questions

REPEATED GAMES

EXAMPLE

- Players play a simultaneous move game repeatedly over time
- If there is a final period: finitely repeated game
- If there is no definite end period: infinitely repeated game
  - players do not know when the game will end but assign some probability to the event that this period could be the last one
- Today's payoff of \$1 is more valuable than tomorrow's \$1
  - This is known as discounting
  - Denote the discount factor by $\delta \in (0, 1)$
  - In PV interpretation: if interest rate is r then δ is;

$$\delta = \frac{1}{1 + r}$$

PAYOFFS, AND REPEATED GAME STRATEGIES

EXAMPLE

- If starting today a player receives an infinite sequence of payoffs

$$u_0, u_1, u_2, u_3, \ldots$$

- The payoffs' present value is

$$u_0 + \delta u_1 + \delta^2 u_2 + \delta^3 u_3, \ldots$$

- For a moment assume that $u_t = u_s$ for all t and s. Then $P.V(u) = u_0/(1 - \delta)$

BACKWARD INDUCTION: EXAMPLE

EXAMPLE

		Player B	
		High	Low
Player A	High	2, 2	0, 3
	Low	3, 0	1, 1

TABLE: No Pure Nash Strategy for this subgame

- **Rule of the game: *Tit-for-Tat***
  - Start with High
  - Play what your opponent played last period
- There are potentially two types of histories
  - Histories in which everybody always played High
    $P.V(u) = 2/(1-\delta)$
  - Histories in which somebody played Low in some period
    $P.V(u) = 3 + 1/(1-\delta)$

PAYOFFS, AND REPEATED GAME STRATEGIES

EXAMPLE

- When does a firm deviate from *High* to *Low*?
- It depends on the market interest rate.

$$2/(1-\delta) < 3 + 1/(1-\delta)$$

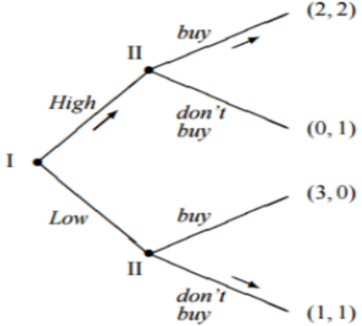
which gives us $\delta < \frac{2}{3}$, and in turn $r > \frac{1}{2}$.

BACKWARD INDUCTION: MORE EXAMPLES

BACKWARD INDUCTION: EXAMPLE

EXAMPLE

- **Player I** is an internet service provider and **player II** a potential customer. They consider entering into a contract of service provision for a period of time.
- The provider decides between two levels of quality of service: **High** or **Low**



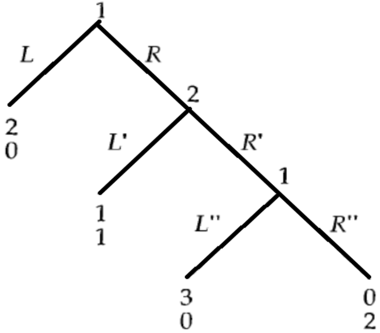
BACKWARD INDUCTION: MORE EXAMPLES

EXAMPLE

- The buyer decides between two actions: **to buy** or **not to buy**
- The service provider, **player I**, makes the first move, choosing **High** or **Low** quality of service. Then the customer, **player II**, is informed about that choice.
- Player II can then decide separately between buy and don't buy in each case.

EXAMPLE

- Consider the following three-move game, in which player 1 moves twice:



- To find the backwards-induction outcome, we begin at the player l' s second move.

EXAMPLE (CONT.)

- Here player 1 faces a choice between a payoff of 3 from L" and a payoff of 0 from R", so L" is optimal.

- Thus, at the second stage, player 2 anticipates that if the game reaches the third stage (**the pen–terminal node**) then 1 will play L", which would yield a payoff of 0 for player 2.

- The second-stage choice for player 2 therefore is between a payoff of 1 from L' and a payoff of 0 from R', so L' is optimal.

- At the first stage, player 1 anticipates that if the game reaches the second stage then 2 will play L', which would yield a payoff of 1 for player 1.

EXPLORING THE RATIONALITY IN BACKWARDS-INDUCTION

- The first-stage choice for player 1 therefore is between a payoff of 2 from L and a payoff of 1 from R, so L is optimal.
- This sequence of arguments establishes that the backwards-induction outcome of this game is for player 1 to choose L in the first stage.
- What would happen if the game did not end in the first stage?

SUBGAME PERFECTION

- Here is a game with 3 subgames.