Dynamic Games: WPBE

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May 5, 2020

Outline

Repeated Games

Example

- Many interactions in the real world have an ongoing structure
  - Firms compete over prices or capacities repeatedly
- In such situations players consider their long-term payoffs in addition to short-term gains
- This might lead them to behave differently from how they would in one-shot interactions
- Consider the following pricing game in the DRAM chip industry

Player B

Player A		High	Low
	High	2, 2	0, 3
	Low	3, 0	1, 1

Table: One Pure Nash Strategy for this sub-game, (*Low, Low*)

Dynamic Rivalry

Example

- If a firm cuts its price today to steal business, rivals may retaliate in the future, nullifying the benefits of the original price cut
- In some concentrated industries prices are maintained at high levels
  - U.S. steel industry until late 1960s
  - U.S. cigarette industry until early 1990s
- In other similarly concentrated industries there is intense price competition
  - Costa Rican cigarette industry in early 1990s
  - U.S. airline industry in 1992
- When and how can firms sustain collusion?
- They could formally collude by discussing and jointly making their pricing decisions
  - Illegal in most countries and subject to severe penalties

Implicit Collusion

Example

- Could firms collude without explicitly fixing prices?
- There must be some reward/punishment mechanism to keep firms in line
- Repeated interaction provides the opportunity to implement such mechanisms
- For example Tit-for-Tat Pricing: mimic your rivals last period price
- A firm that contemplates undercutting its rivals faces a trade-off
  - short-term increase in profits
  - long-term decrease in profits if rivals retaliate by lowering their prices

Implicit Collusion

Example

- Depending upon which of these forces is dominant collusion could be sustained
- What determines the sustainability of implicit collusion?
- Repeated games is a model to study these questions

Repeated Games

Example

- Players play a simultaneous move game repeatedly over time
- If there is a final period: finitely repeated game
- If there is no definite end period: infinitely repeated game
  - players do not know when the game will end but assign some probability to the event that this period could be the last one
- Todays payoff of \$1 is more valuable than tomorrows \$1
  - This is known as discounting
  - Denote the discount factor by $\delta \in (0, 1)$
  - In PV interpretation: if interest rate is r then δ is;

$$\delta = \frac{1}{1 + r}$$

Payoffs, and Repeated Game Strategies

Example

- If starting today a player receives an infinite sequence of payoffs

$$u_0, u_1, u_2, u_3, \ldots$$

- The payoffs' present value is

$$u_0 + \delta u_1 + \delta^2 u_2 + \delta^3 u_3, \ldots$$

- For a moment assume that $u_t = u_s$ for all t and s. Then $P.V(u) = u_0/(1-\delta)$

Backward Induction: Example

Example

Player A		Player B: High	Player B: Low
	High	2, 2	0, 3
	Low	3, 0	1, 1

Table: No Pure Nash Strategy for this subgame

- **Rule of the game: *Tit−for−Tat***
  - Start with High
  - Play what your opponent played last period
- There are potentially two types of histories
  - Histories in which everybody always played High
    $P.V(u) = 2/(1-\delta)$
  - Histories in which somebody played Low in some period
    $P.V(u) = 3 + 1/(1-\delta)$

Payoffs, and Repeated Game Strategies

Example

- When does a firm deviate from *High* to *Low*?
- It depends on the market interest rate.

$$2/(1-\delta) < 3 + 1/(1-\delta)$$

which gives us $\delta < \frac{2}{3}$, and in turn $r > \frac{1}{2}$.

Introduction

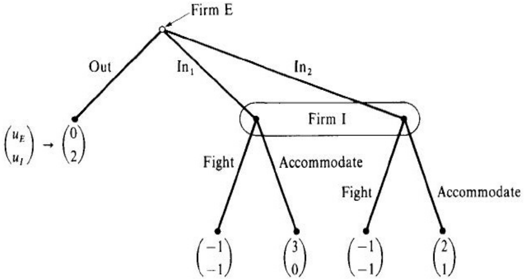
- So far, we have studied games with Nash (Pure or Mixed) Equilibria, games with Bayesian Perfect Equilibria, and Subgame-Perfect Nash Equilibria.

Information		Timing: Simultaneous	Timing: Sequential
	Complete	**Nash**	**Sub-game Perfect Nash Equilibrium**
	Incomplete	**Bayesian Nash**	**Weak Perfect Bayesian Equilibrium**

- This part is going to address Weak Perfect Bayesian Equilibrium.

Systems of Beliefs and Sequential Rationality

- Many games do not have proper sub-game, The **SPNE** concept may fail to insure sequential rationality.



Systems of Beliefs and Sequential Rationality

- The sub-game perfection is powerless in dynamic games where there are no proper sub-games.
- But, still we can find Nash Equilibria in the games of under study

I

		Fight if entry occurs	Acc if entry occurs
E	Out	$\underline{0}$, $\underline{2}$	$\underline{0}$, $\underline{2}$
	In1	-1, -1	$\underline{3}$, $\underline{0}$
	In2	-1, -1	2, $\underline{1}$

Systems of Beliefs and Sequential Rationality

- The game has three Nash equilibria

- We need a theory of "reasonable" choices by players at all nodes, and not just at those nodes that are parts of proper sub-games.

- One way to approach this problem in the above example is to ask: Could **Fight** be optimal for Firm **I** when it must actually act for any belief that it holds about whether Firm **E** played In_1 or In_2? Clearly, no!
- Assume that the Firm **I** believes that Firm **E** plays Action In_1 with probability μ and the In_2 with $(1-\mu)$, **the system of beliefs**.
- $-1 < 1 - \mu$ for all $\mu \in [0, 1]$

Systems of Beliefs and Sequential Rationality

- This motivates a formal development of beliefs in extensive form games.
- To express the notion formally, we need to define two basic components
  1. System of beliefs
  2. Sequential rationality of strategies

Definition

A system of beliefs is a mapping $\mu : \chi \to [0, 1]$ such that, for all $x \in H$, $\Sigma_{x \in H} \mu(x) = 1$.

- In words, a system of beliefs, μ, specifies the relative probabilities of being at each node of an information set, for every information set in the game.

Systems of Beliefs and Sequential Rationality

- Let $E[u_i|H, \mu, \sigma_i, \sigma_{-i}]$ denote player i's expected utility starting at her information set H if her beliefs regarding the relative probabilities of being at any node, $x \in H$ is given by $\mu(x)$, and she follows strategy σ_i while the others play the profile of strategies σ_{-i}.
  - $H = \{In_1, In_2\}$
  - $\sigma_I =$(*Acc if entry occurs*) and $\tilde{\sigma}_I =$(*Fight if entry occurs*)
  - $I = \iota(H)$
  - $E = -\iota(H)$

Definition (sequentially rational at an Info set)

A strategy profile, σ, is sequentially rational at information set H, given a system of beliefs μ, if

$$E[u_{\iota(H)}|H, \mu, \sigma_{\iota(H)}, \sigma_{-\iota(H)}] \geqslant E[u_{\iota(H)}|H, \mu, \tilde{\sigma}_{\iota(H)}, \sigma_{-\iota(H)}]$$

for all $\tilde{\sigma}_{\iota(H)} \in \Delta(S_{\iota(H)})$

Systems of Beliefs and Sequential Rationality

- A strategy profile $\sigma = (\sigma_1, ..., \sigma_I)$ is sequentially rational if no player finds it worthwhile to revise her strategy, given a system of beliefs and her rivals' strategies.

- With these two notions, we are ready to formally define the **WPBE** which involves in two conditions:
  1. Strategies must be sequentially rational given beliefs
  2. Whenever possible, beliefs must be consistent with the strategies.

- In an equilibrium, players should have correct beliefs about their opponents' choices

Example (consistency of beliefs)

- Consider a special case in which each player in her information set mixes among her actions, by $p(out) \geqslant 0$, $p(In_1) \geqslant 0$, $p(In_2) \geqslant 0$ and $p(out) + p(In_1) + p(In_2) = 1$.
- Then consistency of beliefs in $H = \{In_1, In_2\}$ requires that $\mu(In_1) = \frac{p(In_1)}{p(In_1)+p(In_2)}$.
- In which $p(In_1) + p(In_2)$ is the probability of reaching to H, and $p(In_1)$ is probability of reaching to the node following action In_1.

Weak Perfect Bayesian Equilibrium

Definition (WPBE)

A profile of strategies, σ, and a system of beliefs, μ, is a Weak Perfect Bayesian Equilibrium **WPBE**, (σ, μ), if:

1. σ is sequentially rational given μ
2. μ is derived from σ through Bayes rule whenever possible.

That is, for any information set H such that $P(H|\sigma) > 0$, and any $x \in H$,

$$\mu(x) = P(x|\sigma)/P(H|\sigma)$$

Weak Perfect Bayesian Equilibrium

Example (Entery game, Cont.)

- Firm **I** must play ***Acc*** *in Firm **E** enters* in any **WPBE**, because, $-1 < (1 - \mu)$.
- If Firm **I** chooses to play "**Acc**", then best response of **E** is "**Enter**".
- Because $3 > 0$ and $2 > 0$
- What about the (*In_1, "**Acc**" if entry occurs*)? This strategy profile is a part of a **WPBE**.
- The **I**'s beliefs must assign Probability 1 to being at the left node of its information set.
- This strategies are sequentially rational given this system of beliefs and unique **WPBE**.