Generalized Instrumental Variables Estimation of Nonlinear Rational Expectations Models (September, 1982)

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Abstract

- This paper describes a method for estimating and testing **nonlinear rational expectations** models directly from stochastic Euler equations. The estimation procedure makes sample counterparts to the population orthogonality conditions implied by the economic model close to zero. An attractive feature of this method is that the parameters of the dynamic objective functions of economic agents can be estimated without explicitly solving for the stochastic equilibrium.

Rational Expectations Models
(General framework)

- Dividing model variables into state variables(an explicit and known updating rule) and response variables(satisfy an equilibrium relationship).
- The expectation being taken with respect to i.i.d shocks.

Introduction

- Linear-quadratic models → Complete characterizations of the equilibrium → Time series econometric tools
- Alternative nonquadratic objective functions
- Closed-form solutions for the equilibrium time paths of the variables of interest have been obtained only after imposing strong assumptions on the stochastic properties of the "forcing variables," the nature of preferences, or the production technology.

The purpose of this paper…

- An econometric estimation strategy
- Without the theoretical requirement of an explicit representation of the stochastic equilibrium
- Identification and estimation of parameters of economic agents' dynamic (nonquadratic) objective functions
- Only a subset of the economic environment is specified

Basic idea

- The dynamic optimization problems of economic agents
- A set of stochastic Euler equations that must be satisfied in equilibrium
- A set of population orthogonality conditions that depend in a nonlinear way on variables observed
- We construct nonlinear instrumental variables estimators for these parameters by making sample versions of the orthogonality conditions close to zero according to a certain metric.
- Consistent and have a limiting normal distribution under fairly weak assumptions
- An alternative, approximate maximum likelihood procedure, Fair & Taylor

The implications of rational expectations models used in constructing estimators

- $E_t h(x_{t+n}, b_0) = 0$
- E_t denotes both the mathematical conditional expectation and agents' subjective expectations as of date t.

$$Max\ E_t \sum_{t=0}^{\infty} \beta^t U(C_t)$$

$$s.t\ C_t + \sum_{j=0}^{N} P_{jt} Q_{jt} \leq \sum_{j=0}^{N} R_{jt} Q_{jt-M_j} + W_t$$

Q_{jt} : The quantity of asset j held at the end of date t
M_j : The maturity of asset j
P_{jt} : The price of asset j at date t
R_{jt} : The date t payoff from asset purchased at date t-M_j
W_t : Real labor income at date t

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- *Euler's equation:*
- $C_t + \sum_{j=0}^{N} P_{jt}Q_{jt} \leq \sum_{j=0}^{N} R_{jt}Q_{jt-M_j} + W_t$
- $P_{jt}\, U'(C_t) = \beta^{M_j}\, E_t[R_{jt+M_j}U'\left(C_{t+M_j}\right)]$
- $R_{jt+M_j}=P_{jt+M_j}+D_{jt+M_j}$

Constructing h function

- $E_t\left[\beta^{M_j}\frac{U'\left(C_{t+M_j},\gamma\right)}{U'(C_t,\gamma)}x_{jt+M_j}-1\right]=0$
- $x_{jt+M_j}={}^{R_{jt+M_j}}/_{P_{jt}}$

Constructing h function

- $$h(x_{t+n}, b_0) = \begin{bmatrix} \beta^{n_1} \frac{U'(C_{t+n_1},\gamma)}{U'(C_t,\gamma)} x_{1t+n_1} - 1 \\ \vdots \\ \beta^{n} \frac{U'(C_{t+n},\gamma)}{U'(C_t,\gamma)} x_{mt+n} - 1 \end{bmatrix}$$
- $u_{t+n} = h(x_{t+n}, b_0)$

Covariance matrix of u_t

- The matrix E$u_t u_t$' is assumed to have full rank.
- If the m assets are stocks and n1 through n equal unity, then u_t is serially uncorrelated.

Estimation

- We want to estimate the vector b_0 using a generalized instrumental variables procedure.
- Our estimation strategy is to use the theoretical economic model to generate a family of orthogonality conditions.
- Construct a criterion function whose minimizer is our estimate of b_0.
- Our parameter estimator is consistent , asymptotically normal and has an asymptotic covariance matrix that can be estimated consistently.

Equation

- $E_t[u_{t+n}] = 0$
- $E_t[f(x_{t+n}, z_t, b_0)] = 0$
- $f(x_{t+n}, z_t, b_0) = h(x_{t+n}, b_0) \otimes z_t$
- $R^k \times R^q \times R^l \rightarrow R^{mq}$
- $g_0(b) = E_t[f(x_{t+n}, z_t, b_0)]$
- $g_T(b) = \frac{1}{T}\sum_{t=1}^{T} f(x_{t+n}, z_t, b_0)$
- $J_T(b) = g_T(b)' \, W_T g_T(b)$

Most Efficient Estimator

- $D_0 = E[\frac{\partial h}{\partial b}(x_{t+n}, b_0) \otimes z_t]$
- $S_0 = \sum_{j=-n+1}^{n-1} E[f(x_{t+n}, z_t, b_0) f(x_{t+n-j}, z_{t-j}, b_0)']$

Most Efficient Estimator

- $D_T = \frac{1}{T}\sum_{t=1}^{T}\frac{\partial h}{\partial b}(x_{t+n}, b_T) \otimes z_t$
- $R_T(j) = \frac{1}{T}\sum_{t=1+j}^{T} f(x_{t+n}, z_t, b_T) f(x_{t+n-j}, z_{t-j}, b_T)'$
- $W_T^* = \{R_T(0) + \sum_{j=1}^{n-1}[R_T(j) + R_T(j)']\}^{-1}$

Comparison to maximum likelihood

- Amemiya : Maximum likelihood estimators will, in general, be asymptotically more efficient than nonlinear instrumental variables procedures if the distributional assumptions are specified correctly.
- MLE may fail to be consistent if the distribution of the observable variables is misspecified.
- From the first-order conditions of the likelihood function, it can be seen that the method of maximum likelihood implicitly uses the logarithmic orthogonality conditions. The validity of these orthogonality conditions is crucially dependent on distribution of x.

Comparison to maximum likelihood

- $U(C_t) = \frac{(C_t)^{\gamma}}{\gamma}, \gamma < 1$
- $U'(C_t) = (C_t)^{\alpha}, \alpha \equiv \gamma - 1$
- $E_t\left[\beta (x_{kt+1})^{\alpha} x_{jt+1}\right] = 1$
- Stochastic process x is lognormality distributed:
- $U_{jt+1} = \log \beta + \alpha X_{kt+1} + X_{jt+1} + \frac{(\alpha^2 \sigma_{kk} + \sigma_{jj} + 2\alpha\sigma_{kj})}{2}$
- $E\left[U_{jt+1}\right] = 0; E\left[U_{jt+1} X_{t-s}\right] = 0$

Empirical Results

- *NDS (Nondurables plus Services) & ND(Nondurables)*
- $x'_{t+1} = \left[\frac{P_{1t+1}+D_{1t+1}}{P_{1t}} \frac{C_{t+1}}{C_t}\right]$
- $h(x_{t+1}, b_0) = \beta(x_{2t+1})^{\alpha} x_{1t+1} - 1$

Empirical Results

TABLE I

INSTRUMENTAL VARIABLE ESTIMATES FOR THE PERIOD 1959:2–1978:12

Cons	Return	NLAG	$\hat{\alpha}$	$\widehat{SE}(\hat{\alpha})$	$\hat{\beta}$	$\widehat{SE}(\hat{\beta})$	χ^2	DF	Prob
NDS	EWR	1	−.9457	.3355	.9931	.0031	4.9994	1	.9746
NDS	EWR	2	−.9281	.2729	.9929	.0031	7.5530	3	.9438
NDS	EWR	4	−.7895	.2527	.9925	.0031	9.1429	7	.7574
NDS	EWR	6	−.8927	.2138	.9934	.0030	15.726	11	.8484
NDS	VWR	1	−.9001	.3130	.9979	.0025	1.1547	1	.7174
NDS	VWR	2	−.8133	.2298	.9981	.0025	3.2654	3	.6475
NDS	VWR	4	−.6795	.1855	.9973	.0024	6.3527	7	.5008
NDS	VWR	6	−.7958	.1763	.9980	.0023	14.179	11	.7767
ND	EWR	1	−.9737	.1245	.9922	.0031	5.9697	1	.9854
ND	EWR	2	−.9664	.1074	.9919	.0031	8.9016	3	.9694
ND	EWR	4	−.9046	.0926	.9918	.0031	11.084	7	.8650
ND	EWR	6	−.9466	.0793	.9422	.0030	15.663	11	.8459
ND	VWR	1	−.8985	.1057	.9971	.0025	1.5415	1	.8756
ND	VWR	2	−.8757	.0856	.9974	.0025	3.2654	3	.6475
ND	VWR	4	−.8174	.0742	.9967	.0024	7.8776	7	.5008
ND	VWR	6	−.8514	.0629	.9973	.0024	14.938	11	.8147

Empirical Results

TABLE II

MAXIMUM LIKELIHOOD ESTIMATES FOR THE PERIOD 1959:2–1978:12

	Nondurables Plus Services		Nondurables	
	Equally Weighted Returns	Value Weighted Returns	Equally Weighted Returns	Value Weighted Returns
$\hat{\alpha}$	−.5194 (.4607)	−.9349 (.3341)	−.7643 (.2203)	−.9327 (.1579)
$\hat{\beta}$	.9957 (.0037)	.9995 (.0028)	.9957 (.0036)	.9988 (.0028)
χ^2	12.5854	17.8716	14.4456	18.8846
DF	11	11	11	11
Prob.	.6787	.9154	.7907	.9368

Empirical Results

TABLE III

INSTRUMENTAL VARIABLES ESTIMATION WITH MULTIPLE RETURNS

Cons.	NLAG	$\hat{\alpha}$	$\widehat{SE}(\hat{\alpha})$	$\hat{\beta}$	$\widehat{SE}(\hat{\beta})$	χ^2	DF	Prob.
Equally- and Value-Weighted Aggregate Returns 1959:2–1978:12								
NDS	1	−.6875	.2372	.9993	.0023	17.804	6	.9933
NDS	2	−.3624	.1728	.9995	.0022	24.230	12	.9811
NDS	4	−.3502	.1540	.9989	.0021	39.537	24	.9760
ND	1	−.7211	.0719	.9989	.0023	19.877	6	.9971
ND	2	−.5417	.1298	.9988	.0022	24.421	12	.9822
ND	4	−.5632	.1038	.9982	.0021	40.176	24	.9795

Cons.	NLAG	$\hat{\alpha}$	$\widehat{SE}(\hat{\alpha})$	$\hat{\beta}$	$\widehat{SE}(\hat{\beta})$	χ^2	DF	Prob.
Three Industry-Average Stock Returns 1959:2–1977:12								
NDS	1	−.9993	.2632	.9941	.0028	19.591	13	.8941
NDS	4	−.4600	.1388	.9961	.0024	82.735	49	.9982
ND	1	−.9557	.0898	.9935	.0028	22.302	13	.9491
ND	4	−.8085	.0506	.9962	.0023	82.013	49	.9978

Conclusions

- In this paper we have discussed a procedure for estimating the parameters of nonlinear rational expectations models when only a subset of the economic environment is explicitly specified a priori. We also described how to test the over-identifying restrictions implied by the particular economic model being estimated. The advantages of these procedures are that they circumvent the need for explicitly deriving decision rules, and they do not require the specification of the joint distribution function of the observable variables. The techniques are appropriate for any dynamic model whose econometric implications can be cast in terms of a set of orthogonality conditions. As an application of these procedures, we estimated the parameters characterizing preferences in a model relating the stochastic properties of aggregate consumption and stock market returns.


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