Extensive Games with Perfect Information

A strategic situation in which players play a game over time.

At the point of making the decision, players are informed about the action chosen before by the others.

It is also called Dynamic or Sequential-Move Game.

We use the extensive form representation to study this class of games.
Example: An Entry Game:
Definition (Os 155.1): An extensive game with perfect information consists of

- a set of players \( N \)
- a set of terminal histories \( H \); each terminal history is a sequence of actions which no sequence is a proper subhistory of any other sequence
- a player function \( P(h) \) that corresponds a player for every sequence that is a proper subhistory of some terminal history
- for each player, preferences over the set of terminal histories \( u_i(h) \)
The set of actions available to player specified by $P(h)$:

$$A(h) = \{a : (h, a) \text{ is a history}\}$$

Each history is associated with a node; we call the set of nodes $X$.

$X_i$ specifies the set of nodes when it is player $i$’s turn to move.

$$\bigcap_{i} X_i = \emptyset \quad \text{and} \quad \bigcup_{i} X_i = X$$
Game of perfect information: when a player moves at node $x$; he knows that he is at this node.

Only one player moves at a time, and when a player moves, he knows the actions taken by everyone who has moved before him.

Game is finite if the length of all terminal histories are finite.

Then an extensive form game can be represented by:

$$\Gamma = \langle N, H, P(h), u_i(h); X, A(h) \rangle$$
**Example:** Formulate the following game as an extensive form game

The political figures Rosa and Ernesto each has to choose the location for a party congress. The options are Berlin (B) or Havana (H). They choose sequentially. A third person, Karl, determines who chooses first.

Both Rosa and Ernesto care only about the actions they choose, not about who chooses first.

Rosa prefers the outcome in which both she and Ernesto choose B to that in which they both choose H, and prefers this outcome to either of the ones in which she and Ernesto choose different actions; she is indifferent between these last two outcomes.

Ernesto’s preferences differ from Rosa’s in that the roles of B and H are reversed.

Karl’s preferences are the same as Ernesto’s.
A (pure) strategy for a player in an extensive game $\Gamma$ is a plan that specifies an action at each decision node that belongs to her.

A pure strategy for player $i$ is a function $s_i : X_i \rightarrow A$ satisfying

$$s_i(x) \in A(x)$$

$S_i$ is the set of pure strategies for $i$

If we fix a pure strategy for each player, this is a pure strategy profile $s = (s_1, \ldots, s_n)$
A profile strategy \( s = (s_1, \ldots, s_n) \) fully determines what happens in the game.

If there are no mixed strategies involved, each strategy profile is associated exactly by one of the terminal nodes.

If we allow for mixed strategies, then \( s \) determines a probability distribution over \( H \) gives rise to payoff for each player (clearly in this case the utilities are needed to be vN-M preferences).

\[ \langle N, S_i, u_i \rangle \] is the strategic form representation of \( \Gamma \).
The strategic form representation of the entry game:

<table>
<thead>
<tr>
<th></th>
<th>Incumbent</th>
<th></th>
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</thead>
<tbody>
<tr>
<td></td>
<td>Fight</td>
<td>Accommodate</td>
</tr>
<tr>
<td>Entrant</td>
<td>if E plays In</td>
<td>if E plays In</td>
</tr>
<tr>
<td>In</td>
<td>-3, -1</td>
<td>2, 2</td>
</tr>
<tr>
<td>Out</td>
<td>0, 5</td>
<td>0, 5</td>
</tr>
</tbody>
</table>

We can use Nash equilibrium as a solution concept
Without considering the sequential nature of the game, the game has two NE:

(In, Accommodate if E plays In)

and

(Out, Fight if E plays In)

The second NE is not sequentially rational. In fact, Fight is not a credible treat by the Incumbent.

By sequential rationality we mean that at every point in the game tree, the player’s strategy should specify an optimal action. In order to achieve this we need a concept more powerful than the NE.
Entrant knows, if he plays In, it will be followed by Accommodate from Incumbent.

We can consider a **reduced extensive form** for Entrant:
• Backward Induction

The backward induction is used to implement the idea of sequential rationality in finite games of perfect information.

The procedure is:

1) Determine the optimal action at the final decision nodes (those which will be followed only by the terminal nodes); at these decision nodes a simple single-person decision making is involved in optimal actions.

2) Given these actions, determine the optimal behavior at the penultimate decision nodes.

3) And so on anticipate the optimal actions backward through the tree.
Then player 3’s optimal strategy is:

\[ s_3^* = \begin{cases} 
  X & \text{if player 1 plays A and player 2 plays M} \\
  Y & \text{if player 1 plays A and player 2 plays N} \\
  X & \text{if player 1 plays B and player 2 plays N} 
\end{cases} \]
The reduced form game is:

\[ s_2^* = \begin{cases} M & \text{if player 1 plays A} \\ M & \text{if player 1 plays B} \end{cases} \]
The reduced form game is:

Then player 1’s optimal strategy is:

\[ s_1^* = B \]
Then the backward induction identifies the strategy profile:

\[ s_{BI} = (s_1^*, s_2^*, s_3^*) \]

\[ s_1^* = B \]

\[ s_2^* = \begin{cases} M & \text{if player 1 plays A} \\ M & \text{if player 1 plays B} \end{cases} \]

\[ s_3^* = \begin{cases} X & \text{if player 1 plays A and player 2 plays M} \\ Y & \text{if player 1 plays A and player 2 plays N} \\ X & \text{if player 1 plays B and player 2 plays N} \end{cases} \]

Note that this strategy profile is a NE.

Investigate if the game has other NE. However, this is the only NE which satisfies the sequential rationality.
Proposition: Every finite game of perfect information has a pure strategy NE that can be derived through backward induction. Moreover, if no player has the same payoffs at any two terminal nodes, then there is a unique NE that can be derived through backward induction.
• Sub-Game Perfect Equilibrium

**Definition:** A *subgame* of an extensive form game with complete information is a subset of the game if it has the following properties:

1) It begins with a non terminal node $h$ (history)

2) Contains all the nodes that are successors of this node, and contains only these nodes.
Consider this game:

Let’s have a look at the normal form representation:
| Entrant \ (a_0; a_2 | (In, Acc) ; a_3 | (In, Fight) ) | Incumbent \ (a_1 | (In)) | \( (Out, Acc, Acc) \) | \( (Out, Acc, Fight) \) | \( (Out, Fight, Acc) \) | \( (Out, Fight, Fight) \) | \( (In, Acc, Acc) \) | \( (In, Acc, Fight) \) | \( (In, Fight, Acc) \) | \( (In, Fight, Fight) \) |
|---|---|---|---|---|---|---|---|---|---|---|
| \( (Out, Acc, Acc) \) | Acc | 0, 5 | 0, 5 |
| \( (Out, Acc, Fight) \) | 0, 5 | 0, 5 |
| \( (Out, Fight, Acc) \) | 0, 5 | 0, 5 |
| \( (Out, Fight, Fight) \) | 0, 5 | 0, 5 |
| \( (In, Acc, Acc) \) | Acc | 4, 2 | 2, -3 |
| \( (In, Acc, Fight) \) | 4, 2 | -4, -2 |
| \( (In, Fight, Acc) \) | -3, -2 | 2, -3 |
| \( (In, Fight, Fight) \) | -3, -2 | -4, -2 |
Each Strategy Profile:

\[(a_0, a_2 | \text{(In, Acc)}, a_3 | \text{(In, Fight)}); a_1 | \text{(In)}\]

The game has 2 NE:

\[
((\text{In, Acc, Acc}); \text{Acc})
\]

\[
((\text{In, Acc, Fight}); \text{Acc})
\]

From which the second one does not satisfy the principle of sequential rationality.

So we need to define a new notion of equilibrium.
Definition: A profile of strategies $s^*$ in an extensive form game is a subgame perfect equilibrium if it induces a NE in every subgame of the game.

In other words, there is no subgame which player $i$ can do better by choosing a strategy different from $s_i^*$, given that every other player $j$ plays according to $s_j^*$.

Every subgame perfect equilibrium is a NE.

A subgame perfect equilibrium is a strategy profile that induces a NE in every subgame.

$$P(h) = i$$

$$u_i(O_h(s^*)) \geq u_i(O_h(s_i, s_j^*)) \quad \text{for } \forall s_i$$
Revisit our example:

- Fight
- Accommodate

Incumbent

Fight

Entrant

Entrant

Out

\( \begin{array}{c}
\text{Entrant} \\
\text{Entrant} \\
\text{Entrant} \\
\text{Entrant} \\
\text{Incumbent} \\
\end{array} \)

\( \begin{array}{c}
-4 & 2 & 2 & 4 \\
-2 & -3 & -2 & 2 \\
\end{array} \)
The backward induction procedure may be used to find the SPNE:

The procedure is:

4) Find all the subgames of the game
5) Start from the end of tree and find the NE of each of the final subgames.
6) Form the reduced form game, replace each of the final subgames with one of the NE’s payoffs found for that subgame.
7) Repeat steps 2 and 3 and continue the procedure to determine every move in the game. The final outcome is a SPNE.
8) If none of the subgames had multiple NE, then the SPNE found is unique; otherwise repeat the procedure every time by replacing one of the alternative NE.
**Proposition:** The set of subgame perfect equilibria of a finite horizon extensive game with perfect information is equal to the set of strategy profiles isolated by the procedure of backward induction.

**Proposition:** Every finite game of perfect information has a pure strategy SPNE. Moreover, if no player has the same payoffs at any two terminal nodes, then there is a unique SPNE.
Ultimatum game

Pie of size $10

player 1 (proposer) must offer $x to player 2
x between 0 and 10, min unit of one cent

Player 2 (responder) can say YES or NO.
If YES, 2 gets x; 1 gets 10-x
If NO, both get 0

Assume that players only care about their own monetary payoff.
Backwards induction:

If \( x > 0 \); strictly optimal for 2 to play YES
If \( x = 0 \); 2 indifferent between YES and NO.

One backward induction strategy for 2:

YES if \( x > 0 \); NO if \( x = 0 \):

Given this, optimal for 1 to choose \( x = 0.01 \)

Player 1 has almost all the bargaining power, gets almost everything.

Is there another backward induction solution?
Nash equilibria of the ultimatum game:

Can we have any NE that supports any $x^* > 0$

1: offer $x^*$

2: accept any offer $x$ if $x \geq x^*$
   reject any offer strictly less than $x^*$

2's strategy is an incredible threat.

No Nash equilibrium where we observe offers that are actually made being rejected.
Experimental evidence:

Experimental research found that

Player 1 offers significant amounts 35-50% of the pie usually
Player 2 rejects small offers.

Explanations:

2 may be equity conscious (dislike being treated unfairly) therefore may reject low offers

1 may be equity conscious (dislike treating 2 unfairly) or may be optimizing monetary payoff given 2 will reject.

Extent of equity consciousness not commonly known, so: some low offers are made that are rejected
Vote Buying

Legislature with $K$ members, $K$ is odd

Two rival bills, $X$ and $Y$

Two interest groups:

$X$ favours $X$; attaches value $V_X$ to $X$ being passed rather than $Y$

$Y$ favours $Y$; value $V_Y$

Legislature has no preferences over $X$ and $Y$ and only concerns with her Monetary payoff
Sequence of the play:

Each group can commit to pay individual legislators based on their vote

1. X announces schedule of contingent payments to each legislator 
   \[ (x_1, \ldots, x_K) \]

2. Y announces schedule 
   \[ (y_1, \ldots, y_K) \]

3. Legislator \( i \) votes for X if \( x_i > y_i \); and votes for Y otherwise (He is paid according to his vote regardless of the outcome of voting).
\[ \alpha_i = 1 \text{ if } i \text{ votes for } X; \text{ otherwise } \alpha_i = 0 \]

Payoff to interest group X is

\[
\pi_X = \begin{cases} 
V_x - \sum_i x_i \alpha_i & \text{if } X \text{ is passed} \\
- \sum_i x_i \alpha_i & \text{if } Y \text{ is passed}
\end{cases}
\]

Payoff to interest group Y is

\[
\pi_Y = \begin{cases} 
- \sum_i y_i (1 - \alpha_i) & \text{if } X \text{ is passed} \\
V_y - \sum_i y_i (1 - \alpha_i) & \text{if } Y \text{ is passed}
\end{cases}
\]
Let $m$ denote the least number of votes necessary to win the vote:

$$m = \frac{K + 1}{2}$$

If $X$ makes an equal payment of $c$ to each legislator, then $X$ can win only if:

$$mc > V_Y$$

Or

$$c > \frac{V_Y}{m}$$
But this must be profitable for X:

\[ V_X \geq Kc \]

\[ V_X \geq K \frac{V_Y}{m} \]

If this condition is satisfied, X pays \( \frac{V_Y}{m} \) to each legislator, and Y pays nothing, and X gets passed.

If not, then X pays nothing, Y pays nothing, and Y gets passed.
Firm and Union:

A firm’s output is

\[ q = \begin{cases} 
L(100 - L) & \text{if } L \leq 50 \\ 
2500 & \text{if } L > 50 
\end{cases} \]

The price of output is 1

1) A union that represents workers presents a wage demand \( \omega \geq 0 \)
2) The firm either accepts or rejects the offer
3) If the firm accepts, it chooses the number \( L \) of workers to employ, and if it rejects the demand, no production takes place \( (L = 0) \).

Firm’s payoff:

\[ \pi_F = q - \omega L \]

Union’s payoff:

\[ \pi_U = \omega L \]
The extensive form representation of the game:
Backward induction

In the last stage: it is clear that the firm will not employ more than 50

\[ \pi_F(L, \omega) = q - \omega L = L(100 - L) - \omega L \]

\[ \frac{\partial \pi_F}{\partial L} = 100 - 2L - \omega = 0 \]

\[ L^* = \frac{100 - \omega}{2} = 50 - \frac{\omega}{2} \leq 50 \]

In the penultimate stage: Firm accepts \( \omega \) if \( \pi_F(L^*, \omega) \geq 0 \)

\[ \pi_F = \left(50 - \frac{\omega}{2}\right)\left(50 + \frac{\omega}{2}\right) - \omega \left(50 - \frac{\omega}{2}\right) = \left(50 - \frac{\omega}{2}\right)^2 \]
Since $\pi_F \geq 0$, then firm accepts any offer.

In the first stage: Union chooses $\omega$ to maximize its payoff:

$$\pi_U = \omega \left( 50 - \frac{\omega}{2} \right)$$

$$\frac{\partial \pi_U}{\partial \omega} = 50 - \omega = 0$$

$$\omega^* = 50$$

Firms accepts this offer and employs $L^* = 25$

$$\pi_U(L^*, \omega^*) = 1250 \quad \text{and} \quad \pi_F(L^*, \omega^*) = 625$$

There are outcomes $(\hat{L}, \hat{\omega})$ of the game which are preferred to the SPNE by both players. They should satisfy:

$$\pi_U(\hat{L}, \hat{\omega}) > 1250 \quad \text{and} \quad \pi_F(\hat{L}, \hat{\omega}) > 625$$
A Patent Race

Model of patent race (R & D competition)

Two players: 1 & 2

Single prize size $V$ goes to winner (first to finish), loser gets 0

Player 1 is $k_1$ steps from finish, 2 is $k_2$ steps

Players move sequentially

A player can move 0; 1 or 2 steps at cost 0, $c(1)$ and $c(2)$
If both players don’t move in consecutive moves, game ends and neither gets prize.

A race is fully specified by $k_1, k_2$ and the identity of the player who moves first.

$G_i(k_1, k_2)$ is the race where it is player $i$'s turn to move.

We use the backward induction to find the SPNE.
Start with $G_1(k_1, k_2)$

After some moves are made in $G_1(k_1, k_2)$

Suppose that it is player $i$'s turn to move; player 1 is $m_1$ steps away and player 2 is $m_2$

We have a new race $G_i(m_1, m_2)$

Backward induction says that the solution of $G_i(m_1, m_2)$ should determine how we solve $G_1(k_1, k_2)$
So let’s start with race with least steps (the game that may be finished by just one move) and use the solution of these to analyze longer races

**Example:** \( V = 7; c(1) = 1; \) and \( c(2) = 4 \)

So if we have only one firm (e.g. firm 2 is 100 steps away and 1 is 5 steps away):

Player 1 will proceed 1 step at a time.

Consider games where 1 has to move, \( G_1(m, n) \)(symmetric argument for games \( G_2(n, m) \))
$G_1(1, n)$: 1 will take one step and win.

$G_1(2,1)$: 1 knows that 2 will win next time unless 1 already wins.
   So 1 will take 2 steps and win, since $7 - 4 > 0$

$G_1(2,2)$: if 1 takes one step, then it knows the game next period is $G_2(1,2)$
   We know from above that 2 will take 2 steps then; 
   So 1 will take 2 steps and win.

$G_1(3,1)$: 1 will take 0 steps since 2 will win next period

$G_1(2,3)$: if 1 takes 1 step, then 2 cannot win in next period;
   So 1 takes 1 step, 2 takes 0 next period, and 1 wins in third period.

$G_1(3,2)$: 1 will take 0 steps; then 2 will take 1 step, and 1 will take 0 steps
   and 2 will win with 1 more steps
Stackelberg Duopoly

Setting similar to the Cournot model but now firms move sequentially (one firm is the leader the other is the follower)

Players: The two firms $N = \{L, F\}$

Terminal histories: The set of all sequences $(q_L, q_F)$ of outputs for the firms $q_L, q_F \geq 0$

Player function: $P(\emptyset) = L$ and $P(q_L) = F; \forall q_L$

Preferences: The profit of firm $i$ to the terminal history $(q_L, q_F)$:
$$q_i \cdot p(q_L + q_F) - C_i(q_i)$$
Solve for the case:

\[ P = \begin{cases} \alpha - Q & \text{if } Q \leq \alpha \\ 0 & \text{if } Q > \alpha \end{cases} \]

\[ C_i(q_i) = cq_i \quad \forall i \]

It can be shown that the SPNE is:

\[ q_L = \frac{1}{2}(\alpha - c) \]

\[ q_F = \max \left\{ 0, \frac{1}{2}(\alpha - c - q_L) \right\} \]

Recall our solution for Cournot model with \( n \) firms:

\[ q_i^* = \frac{1}{n + 1}(\alpha - c) \]
Hold-up Game

It has a similar setting to the ultimatum game, but now Player 2 can affect the size of the cake before the negotiation starts.

2 takes an action (effort; $E$) that affects the size $c$ of the pie to be divided.

Effort levels are $E \in \{L, H\}$ and $H > L$.

She may exert little effort, resulting in a small pie, of size $c_L$.

Or great effort, resulting in a large pie, of size $c_H$.

The effort is costly; her payoff is $x - E$ if her share of the pie is $x$. 