Neoclassical Growth Model

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Neoclasscial Growth Model

- Social Planner problem

$$
\begin{aligned}
V(k_0) &= \max_{\{c_t, i_t\}} \sum_{t=0}^{\infty} \beta^t U(c_t) \\
c_t + i_t &= f(k_t) = A_t k_t^{\alpha} \\
k_{t+1} &= (1-\delta) k_t + i_t
\end{aligned}
$$

- FOCs and the Euler Equation
- The recursive formula using shooting algorithm
- Transition path
- Steady State

Neoclasscial Growth Model

- Dynamic programming framework

$$
\begin{aligned}
V(k) &= \max_{\{c_t, s_t\}} \left\{ U(c) + \beta V(k') \right\} \\
c + i &= f(k) = Ak^{\alpha} \\
k' &= (1-\delta)k + i
\end{aligned}
$$

NGM: Lagrangian

- Social Planner problem

$$
\begin{aligned}
V(k_0) &= \max_{\{c_t, i_t\}} \sum_{t=0}^{\infty} \beta^t U(c_t) \\
c_t + k_{t+1} &= A_t k_t^{\alpha} + (1-\delta) k_t
\end{aligned}
$$

- Lagrangian

$$
\Lambda = \sum_{t=0}^{\infty} \beta^t U(c_t) + \sum_{t=0}^{\infty} \lambda_t \left(A_t k_t^{\alpha} + (1-\delta) k_t - c_t - k_{t+1} \right)
$$

NGM: FOCs

- FOCs

$$\begin{aligned} [c_t] &: \quad \beta^t U'(c_t) = \lambda_t \\ [k_{t+1}] &: \quad \lambda_t = \lambda_{t+1}\left(\alpha A_{t+1} k_{t+1}^{\alpha-1} + (1-\delta)\right) \end{aligned}$$

- Euler Equation

$$\frac{\lambda_t}{\lambda_{t+1}} = \frac{U'(c_t)}{\beta U'(c_{t+1})} = \alpha A_{t+1} k_{t+1}^{\alpha-1} + (1-\delta)$$

NGM: Allocation Solution

- Conditions

$$\begin{aligned}\frac{U'(c_t)}{\beta U'(c_{t+1})} &= \alpha A_{t+1} k_{t+1}^{\alpha-1} + (1-\delta) \\ c_t + k_{t+1} &= A_t k_t^{\alpha} + (1-\delta) k_t\end{aligned}$$

- Solve for c_t, k_{t+1}

NGM with Constant productivity: Steady State

- Steady State (SS) allocation:

$$
\begin{aligned}
c_t &= c \\
k_{t+1} &= k
\end{aligned}
$$

- take $\beta = \frac{1}{1+\rho}$

$$
\begin{aligned}
\frac{U'(c)}{\beta U'(c)} &= 1+\rho = \alpha A k^{\alpha-1} + (1-\delta) \\
k &= \left(\frac{\alpha A}{\rho+\delta}\right)^{\frac{1}{1-\alpha}}
\end{aligned}
$$

NGM with Constant productivity: Steady State

- and

$$
\begin{aligned}
c+k &= Ak^{\alpha}+(1-\delta)k \\
c &= Ak^{\alpha}-\delta k = A^{\frac{1}{1-\alpha}}\left(\left(\frac{\alpha}{\rho+\delta}\right)^{\frac{\alpha}{1-\alpha}}-\delta\left(\frac{\alpha}{\rho+\delta}\right)^{\frac{1}{1-\alpha}}\right) \\
&= A^{\frac{1}{1-\alpha}}\left(\frac{\alpha}{\rho+\delta}\right)^{\frac{1}{1-\alpha}}\left(\frac{\rho+\delta(1-\alpha)}{\alpha}\right)
\end{aligned}
$$

- output

$$
y = Ak^{\alpha} = A^{\frac{1}{1-\alpha}}\left(\frac{\alpha}{\rho+\delta}\right)^{\frac{\alpha}{1-\alpha}}
$$

NGM with Constant productivity: Steady State

- Explain the role of A, k_0, ρ, δ

NGM with Constant productivity: Steady State

- Transitional Dynamics Solutions
  - Approximation methods: log-linearization, Quadratic approximation, ...
  - Numerical methods: Time domain, State domain (DP)

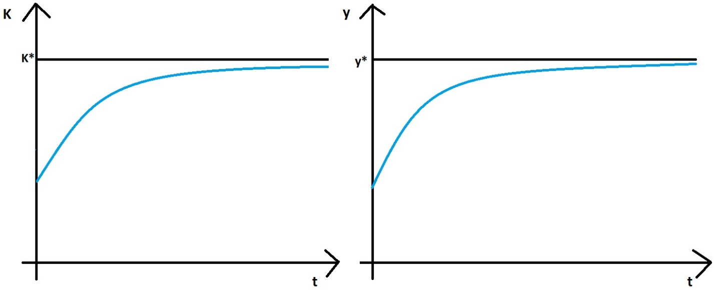
NGM: Transitional Dynamics

- Here, we use Shooting algorithm as a numerical method with time domain
  1. Guess k_1.
  2. Solve for c_t for $t \geq 0$
  3. Solve for k_{t+1} for $t \geq 1$
  4. Iterate 2,3 for T periods.
  5. If converged to the ε neigborhood of the Steady state, stop,
  6. If not, Update your guess for k_1 and go to step 2.
- So for each k_0, we find unique values c_0 and k_1

NGM: Transitional Dynamics

- In other words: For each k, we find unique function $c_0(k)$ and $k_1 = g(k)$.
- Iterating over this function, we find $k_1, k_2, \ldots$ and $c_0, c_1, \ldots$. We converge to the steady state.
- Features and their interpretations:
  - Both are Increasing in k.
  - Both are increasing in A.
  - MPK is decreasing in k.
  - Net Investment is decreasing in k.
  - Speed of growth is decreasing in k.

NGM: Transitional Dynamics



NGM: Transitional Dynamics

- Shock Analysis:
  - Negative shock to k0
  - Permanent shock to A
  - Temporary shock to A

- HH's problem

$$
\begin{aligned}
V(k_0) &= \max_{\{c_t, s_t\}} \sum_{t=0}^{\infty} \beta^t U(c_t) \\
c_t + i_t &= w_t l_t + v_t k_t + \pi_t \\
k_{t+1} &= (1-\delta) k_t + i_t
\end{aligned}
$$

Take $l_t = 1$

- Representative Firm's problem

$$
\max \pi_t = AF(K_t, L_t) - (w_t L_t + v_t K_t)
$$

- Market Clearing

$$
\begin{aligned}
K_t^s &= K_t^d \\
L_t^s &= L_t^d \\
Y_t^s &= Y_t^d = C_t + I_t
\end{aligned}
$$

- In case of homogeneity: $C_t = Nc_t, I_t = Ni_t, L_t = N * 1, K_t = Nk_t$

Representative Agent Model

- HH FOCs

$$\begin{aligned}\frac{U'(c_t)}{\beta U'(c_{t+1})} &= 1-\delta+v_{t+1}\\ c_t+k_{t+1} &= w_t*1+(1-\delta+v_t)k_t+\pi_t\end{aligned}$$

- Firm FOCs

$$\begin{aligned}v_t &= F_k(K_t,L_t)=\alpha AK_t^{\alpha-1}L_t^{1-\alpha}=\alpha Ak_t^{\alpha-1}\\ w_t &= F_L(K_t,L_t)=(1-\alpha)AK_t^{\alpha}L_t^{-\alpha}=(1-\alpha)Ak_t^{\alpha}\\ \pi_t &= 0\\ Y_t &= w_tL_t+v_tK_t \text{ and } y_t=w_t*1+v_tk_t\end{aligned}$$

Representative Agent Model

- Equilibrium conditions:

$$\frac{U'(c_t)}{\beta U'(c_{t+1})} = 1 - \delta + \alpha A k_{t+1}^{\alpha-1}$$

$$c_t + k_{t+1} = A k_t^{\alpha} + (1-\delta) k_t$$

- and prices and other allocations would be

$$v_t = \alpha A k_t^{\alpha-1}$$

$$w_t = (1-\alpha) A k_t^{\alpha}$$

$$y_t = A k_t^{\alpha}$$

$$i_t = k_{t+1} - (1-\delta) k_t$$

Representative Agent Model

- Equilibrium conditions:

$$\begin{aligned}\frac{U'(c_t)}{\beta U'(c_{t+1})} &= 1-\delta+\alpha A k_{t+1}^{\alpha-1} \\ c_t + k_{t+1} &= A k_t^{\alpha} + (1-\delta) k_t\end{aligned}$$

- and prices and other allocations would be

$$\begin{aligned}v_t &= \alpha A k_t^{\alpha-1} \\ w_t &= (1-\alpha) A k_t^{\alpha} \\ y_t &= A k_t^{\alpha} \\ i_t &= k_{t+1} - (1-\delta) k_t\end{aligned}$$

Representative Agent Model: Steady State

- SS

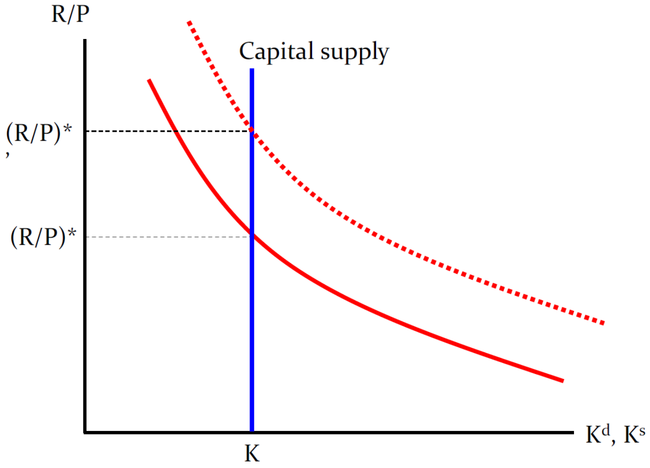
$$
\begin{aligned}
\frac{1}{\beta} &= 1-\delta+\bar{v} \\
\bar{v} &= \rho+\delta \\
\bar{k} &= \left(\frac{\alpha A}{\rho+\delta}\right)^{\frac{1}{1-\alpha}} \\
w &= (1-\alpha)\, A k^{\alpha}
\end{aligned}
$$

Representative Agent Model: Transitional Dynamics

- Similar to the Ramsey Model
- $k_1 = g(k_0)$ ....

Representative Agent Model

- Explanation of Spot and Persistency Effects
- Explanation of Autocorrection and recovery mechanisms



Neoclasscial Growth framework: Constant Growth

- Growth:

$$
\begin{aligned}
V(k_0) &= \max_{\{c_t, s_t\}} \sum_{t=0}^{\infty} \beta^t U(c_t) \\
c_t + i_t &= f(k_t) = A_t k_t^{\alpha} \\
k_{t+1} &= (1-\delta) k_t + i_t \\
A_t &= (1+g) A_{t-1}
\end{aligned}
$$

- Euler Equation
- Balanced Growth Path
- Transition path
- Steady State

- EE:

$$\frac{U'(c_t)}{\beta U'(c_{t+1})} = 1-\delta+\alpha A_{t+1}k_{t+1}^{\alpha-1} = 1-\delta+\alpha v_{t+1}$$

- Balanced Growth Path:

$$\begin{aligned} \frac{c_t^{-\sigma}}{\beta c_{t+1}^{-\sigma}} &= \frac{1}{\beta}\left(\frac{c_{t+1}}{c_t}\right)^{\sigma} = \frac{1}{\beta}(1+g_c)^{\sigma} = (1+\rho)(1+g_c)^{\sigma} \\ v_{t+1} &= (1+\rho)(1+g_c)^{\sigma} - (1-\delta) = \bar{v} \end{aligned}$$

- $\bar{v} = \alpha\frac{\bar{y}}{\bar{k}}$. so y and k grow at the same rate. similarly c and i according to the law of motion and the resource constraint. so $g_y = g_k = g_i = g_c = \bar{g}$

$$\begin{aligned} 1+g_y &= (1+g)(1+g_k)^{\alpha} \\ 1+\bar{g} &= (1+g)^{\frac{1}{1-\alpha}} \text{ or } \bar{g}\tilde{\ }\frac{g}{1-\alpha} \end{aligned}$$

- Explanation: Why it is different from SS.

Neoclasscial Growth framework: Constant Growth

- Scale down with $1 + \bar{g}$
- Then solve for the transitional dynamics.
- Show the graphical representation.